Asymptotic minimization of robust “downside” risk

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Abstract

We consider a stochastic factor model that is robust against model ambiguity by taking into account a whole class of possible prior probabilistic models. Our objective is the long term minimization of robust downside risk, defined as the worst-case probability that the portfolio's growth rate falls below a given target. The asymptotic formulation leads to a large deviations control problem for capacities. By duality this problem is related to the optimal growth rates of robust expected power utility with negative risk aversion parameters. Our main results characterize these dual growth rates in terms of ergodic Bellman equations and establish a duality relation between the primal problem and the asymptotics of robust expected power utility.

Key words: benchmark approach to long term investment, model ambiguity, robust utility maximization, risk-sensitive control, ergodic Bellman equation

Mathematics Subject Classification (2000): 49L20, 60F10, 91B16, 91B28, 93E20

1 Introduction

The classical literature on optimal investment decisions in a financial market usually involves the maximization of a utility functional. Utility maximization, however, is conceptually related to specific numerical representations of the investor's preferences; see, e. g., Föllmer, Schied and Weber [13] and the references therein. For institutional managers utility maximization thus creates severe difficulties. On the one hand, the preferences of their customers and the corresponding numerical representations are not really known exactly. On the other hand, the individual preferences of the managers and of the various customers with shares in the same investment fund will typically be different. This suggests to look for an “intersubjective” criterion for optimal portfolio management which is acceptable for a large variety of investors.

Such an alternative consists in evaluating the performance of the portfolio relative to a given *benchmark* such as a stock index. From the viewpoint of *risk management* it is of particular importance to control the hazard of underperforming a given benchmark. In this paper, we propose a criterion of this type for optimal portfolio management that

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- allows for *model ambiguity*
- and takes a *long term* view.

Here the investor has in mind a collection $\mathcal{Q}$ of possible probability distributions of market events and takes a worst-case approach to cope with model ambiguity. His goal is the minimization of robust "downside" risk for a long term horizon T, defined as the worst-case probability that the portfolio's growth rate L_T^π falls below a target $c \in \mathbb{R}$. In the spirit of *large deviations theory* the asymptotic problem can be formulated as

$$\text{minimizing} \quad \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q\in\mathcal{Q}} Q[L_T^\pi \leq c] \quad \text{among all investment strategies } \pi. \tag{1}$$

A benchmark criterion of this form may be acceptable for a large variety of investors as long as they agree on the choice of the class $\mathcal{Q}$ and the level of satisfaction c, and it may thus be of particular interest for institutional managers with long term horizon, such as mutual fund managers. The asymptotic formulation has the advantage of allowing for stationary optimal policies and is typically more tractable. Moreover, the asymptotic ansatz may provide useful insight for investment decisions with long but finite maturity.

For a specific model Q, the non-robust version of problem (1) has been suggested by Pham [29] as a supplement to his long term outperformance criterion:

$$\text{maximize} \quad \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln Q[L_T^\pi \geq c] \quad \text{among all investment strategies } \pi. \tag{2}$$

The solution to (2) (see [29], Theorem 3.1) is derived in general form by large deviations arguments, similar to the Gärtner-Ellis theorem (see, e. g., [5], Theorem 2.3.6). The dual problem leads to a *risk-sensitive control* problem and can be seen as asymptotic maximization of expected power utility with positive parameters. For a discussion within specific financial market models we refer to Hata and Iida [17], Hata and Sekine [19], and Sekine [33]. For minimizing down-side risk, Pham only gave a heuristic sketch. A rigorous solution via duality was obtained first by Hata, Nagai and Sheu [18] for a linear Gaussian factor model. We are going to adapt their method to a non-linear stochastic factor model and to the *robust* large deviations control problem (1). In that case, the dual problem involves the optimal growth rates of robust expected power utility with negative risk aversion parameters.

The paper is organized as follows: Section 2 describes the model, formulates the problem, and develops the duality ansatz for the asymptotic minimization of robust downside risk. The first step is to analyze the asymptotics of robust expected power utility with negative parameter. In Section 3 we tackle this dual problem by combining the duality approach to robust utility maximization with dynamic programming methods for a varying time horizon. Section 4 contains our duality results for the "robust" large deviations criterion. In Section 5 we illustrate our results with two examples, where explicit solutions can be derived: a Black-Scholes model with uncertain drift and a geometric Ornstein-Uhlenbeck model with uncertain rate of mean reversion.

2 The model and problem formulation

We consider a financial market model with infinite time horizon consisting of two liquidly tradable primary products: one locally riskless asset (*money market account*) and one risky asset (*stock*). Their respective price processes $S^0 = (S_t^0)_{t\geq 0}$

and $S^1 = (S^1_t)_{t\geq 0}$ are defined on the canonical path space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q_0)$ of a two-dimensional Wiener process $W = (W^1_t, W^2_t)_{t\geq 0}$, and they are assumed to be affected by an external "economic factor" process $Y = (Y_t)_{t\geq 0}$, driven by W. The spectrum of possible factors includes dividend yields, short-term interest rates, price-earning ratios, yields on various bonds, the rate of inflation, etc.. In this financial market, an investor faces *model uncertainty*, or *model ambiguity*, in the sense that the dynamics of both the price processes S^0, S^1 and the factor process Y are not precisely known. This model ambiguity will be taken into account by admitting an entire class $\mathcal{Q}$ of probabilistic models, viewed as perturbations of the following reference model Q_0. Under Q_0 the price process of the risky asset evolves according to the SDE

$$dS^1_t = S^1_t(m(Y_t)\,dt + \sigma\,dW^1_t), \tag{3}$$

and the dynamics of the locally riskless asset is given by

$$dS^0_t = S^0_t r(Y_t)\,dt, \quad S^0_0 = 1.$$

Thus the market price of risk is defined via the function

$$\theta(y) := \tfrac{m(y)-r(y)}{\sigma}. \tag{4}$$

The economic factor processes is modeled by the SDE

$$dY_t = g(Y_t)\,dt + \rho\,dW_t = g(Y_t)\,dt + \rho_1\,dW^1_t + \rho_2\,dW^2_t. \tag{5}$$

We suppose that the factor process cannot be traded directly, hence the market model is typically incomplete. Such market models are very popular in mathematical finance and economics; see, e. g., [14], [6], [4], and the references therein.

Throughout this paper, we impose the following general assumptions on the coefficients of the diffusions:

Assumption 2.1. *The short rate function r belongs to $C^2_b(\mathbb{R})$, and g, m admit derivatives $g_y, m_y \in C^1_b(\mathbb{R})$, where $C^k_b(\mathbb{R})$ denotes the class of all bounded functions with bounded derivatives up to order k. Moreover, we assume that $\sigma, \|\rho\| > 0$.*

Here $\|\cdot\|$ denotes the Euclidian norm in $\mathbb{R}^2$, and we will use $(\cdot,\cdot)$ to indicate the associated inner product. In particular, Assumption 2.1 ensures that g and θ grow at most linearly.

In reality, however, the "true" price dynamics is not really known exactly; in particular the drift terms appearing in (3) and (5) are subject to model ambiguity. To cope with the uncertainty in the choice of the drifts, we consider the parameterized class of possible probabilistic models

$$\mathcal{Q} := \{Q^\eta | \eta = (\eta_t)_{t\geq 0} \in \mathcal{C}\}, \tag{6}$$

on $(\Omega, \mathcal{F})$, where $\mathcal{C}$ denotes the set of all progressively measurable processes $\eta = (\eta_t)_{t\geq 0}$ such that $\eta_t = (\eta^{11}_t, \eta^{12}_t, \eta^{21}_t, \eta^{22}_t)$ belongs $dt \otimes Q_0$-a. e. to some fixed compact and convex set $\Gamma \subset \mathbb{R}^4$ which contains the origin. More precisely, for $\eta \in \mathcal{C}$ and any fixed horizon T, the restriction of Q^η to the σ-field $\mathcal{F}_T$ is specified by the Radon-Nikodým density

$$D^\eta_T := \tfrac{dQ^\eta}{dQ_0}\big|_{\mathcal{F}_T} := \mathcal{E}\Big(\int_0^\cdot \eta^{1\cdot}_t Y_t + \eta^{2\cdot}_t\,dW_t\Big)_T \tag{7}$$

with respect to the reference measure Q_0. Here $\mathcal{E}(\cdot)$ is the Itô exponential. In particular, we have $Q_0 \in \mathcal{Q}$, and any measure $Q^\eta \in \mathcal{Q}$ is locally equivalent to Q_0. Moreover, it follows as in [20], Lemma 3.1, that the set $\mathcal{Q}$ is convex. By Girsanov's theorem, the processes S^1, Y evolve under Q^η according to the SDEs

$$dY_t = [g(Y_t) + (\rho, \eta_t^{1\cdot} Y_t + \eta_t^{2\cdot})]\,dt + \rho\,dW_t^\eta, \tag{8a}$$

$$dS_t^1 = S_t^1([m(Y_t) + \sigma(\eta_t^{11} Y_t + \eta_t^{21})]\,dt + \sigma\,dW_t^{1,\eta}), \tag{8b}$$

where W^η is a two-dimensional Q^η-Wiener process. Each model $Q^\eta \in \mathcal{Q}$ thus corresponds to an affine perturbation of the drifts in our reference model Q_0. Special cases of the present "robust" market model are the *Black-Scholes model with uncertain drift* and a *geometric Ornstein-Uhlenbeck model with uncertain rate of mean reversion*; see Section 5.

Let us now describe our benchmark approach to optimal long term investment in the face of model ambiguity: We consider an investor with initial capital $x_0 > 0$ who wants to invest at any time t a proportion π_t of the current wealth X_t^π into the risky asset S^1. The remaining wealth $(1-\pi_t)X_t^\pi$ is always put into the money market account S^0. Then his wealth generated by the trading strategy $\pi = (\pi_t)_{t\geq 0}$ evolves according to the SDE

$$\begin{aligned} dX_t^\pi &= X_t^\pi(\pi_t \tfrac{dS_t^1}{S_t^1} + (1-\pi_t)\tfrac{dS_t^0}{S_t^0}) \\ &= X_t^\pi(r(Y_t)\,dt + \pi_t\sigma[(\theta(Y_t) + \eta_t^{11} Y_t + \eta_t^{21})\,dt + dW_t^{1,\eta}]) \end{aligned} \tag{9}$$

with initial condition $X_0^\pi = x_0$. In other words, the restructuring of the portfolio is self-financing in the sense that any change in the portfolio value equals the profit or loss due to changes in the asset prices. For convenience we omit the explicit dependence of X^π on the initial capital x_0, since it will be irrelevant for our purpose of long term investment. We say that a progressively measurable process π is admissible up to maturity T if (9) admits a unique, strong, and almost surely positive solution $(X_t^\pi)_{t\in[0,T]}$. A strategy π will be called admissible if it is admissible for all $T > 0$. We shall denote by $\mathcal{A}_T$ the class of all T-admissible strategies π, and by $\mathcal{A}$ the class of all strategies which are admissible.

We suppose that the investor has in mind a level of return c and wants to avoid that the portfolio's growth rate

$$L_T^\pi := \tfrac{1}{T}\ln X_T^\pi$$

falls below the threshold c in the long run, at least with maximal probability. But instead of a fixed probabilistic model the investor takes into account the whole class $\mathcal{Q}$ in (6) and chooses a worst-case approach to evaluate the probability that his portfolio underperforms a virtual savings account with interest rate c. For a finite horizon T, this corresponds to

$$\text{minimizing} \quad \sup_{Q^\eta\in\mathcal{Q}} Q^\eta[L_T^\pi \leq c] \quad \text{among all } \pi \in \mathcal{A}_T. \tag{10}$$

But what happens in the long run? If the growth rates L_T^π satisfy under $Q^\eta \in \mathcal{Q}$ a *large deviations principle* with rate function I^π, then $Q^\eta[L_T^\pi \leq c] \approx \exp(-I^\pi(c)T)$ as $T \uparrow \infty$. Clearly, the larger is $I^\pi(c)$, the more chance there is of realizing a growth rate L_T^π above c asymptotically. The long term view thus amounts to maximizing the rates $I^\pi(c)$, or equivalently to

$$\text{minimizing} \quad \varliminf_{T\uparrow\infty} \tfrac{1}{T}\ln Q^\eta[L_T^\pi \leq c] \quad \text{among all } \pi \in \mathcal{A}. \tag{11}$$

This suggests the following asymptotic formulation of the robust problem (10):

$$\text{minimize} \quad \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} Q^\eta[L_T^\pi \le c] \quad \text{among all } \pi\in\mathcal{A}. \tag{12}$$

The asymptotic solution should provide good insight for the optimal investment criterion (10) with long but finite horizon. From a mathematical point of view, (12) can be seen as large deviations control problem for the capacity $\sup_{Q^\eta\in\mathcal{Q}} Q^\eta[\cdot]$; see, e. g., Hu [21] and Gao [15] for recent extensions of Cramér's and Sanov's theorem to capacities.

Its non-robust version (11) has been proposed by Pham [29], Section 6, as a counterpart to his outperformance criterion (2). He suggests that a solution can be derived as follows: Suppose that the logarithmic moment generating function

$$\Lambda_{Q^\eta}(\lambda,\pi) := \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln E_{Q^\eta}[e^{\lambda T L_T^\pi}]$$

of the process L^π fulfills the conditions required by the Gärtner-Ellis theorem; see, e. g., [5], Theorem 2.3.6. Then $(L_T^\pi)_{T\ge 0}$ satisfies a *large deviations principle*, and the large deviations probability of downside risk with respect to π should be measured by the Fenchel-Legendre transform of the function $\Lambda_{Q^\eta}(\cdot,\pi)$:

$$\varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln Q^\eta[L_T^\pi \le c] = -\sup_\lambda\{\lambda c - \Lambda_{Q^\eta}(\lambda,\pi)\}.$$

Taking the infimum among all $\pi\in\mathcal{A}$, it is thus natural to expect that the rate

$$\underline{J}_{Q^\eta}(c) := \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln Q^\eta[L_T^\pi \le c], \quad c\in\mathbb{R}, \tag{13}$$

is described by the Fenchel-Legendre transform $\underline{\Lambda}^*_{Q^\eta}(c) := \sup_\lambda\{\lambda c - \Lambda_{Q^\eta}(\lambda)\}$ of

$$\Lambda_{Q^\eta}(\lambda) := \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln E_{Q^\eta}[e^{\lambda T L_T^\pi}].$$

However, Pham provided only heuristic arguments. His conjecture has been verified by Hata, Nagai and Sheu [18] in the special case of a linear Gaussian factor model.

In this paper, we are going to extend their results to the robust case (12) and to a stochastic factor model with non-linear coefficients. For this purpose, we denote the optimal rate of decay for robust downside risk by

$$\underline{J}(c) := \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} Q^\eta[L_T^\pi \le c], \quad c\in\mathbb{R}. \tag{14}$$

In spirit of the Gärtner-Ellis theorem, our main result will state a duality relation between the value function $\underline{J}$ and the rates

$$\Lambda(\lambda) = \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[e^{\lambda T L_T^\pi}]. \tag{15}$$

For this purpose, we consider the Fenchel-Legendre transform

$$\underline{\Lambda}^*(c) := \sup_{\lambda<0}\{\lambda c - \Lambda(\lambda)\}, \quad c\in\mathbb{R}, \tag{16}$$

that is convex, nonincreasing, and lower-semicontinuous on $\mathbb{R}$.

Proposition 2.1. *For any $c\in\mathbb{R}$, $\underline{J}(c) \le -\underline{\Lambda}^*(c)$.*

Proof. For all $\pi \in \mathcal{A}$, $Q^\eta \in \mathcal{Q}$, and $\lambda < 0$ Tchebychev's inequality yields

$$Q^\eta[L_T^\pi \leq c] = Q^\eta[e^{\lambda T L_T^\pi} \geq e^{\lambda T c}] \leq e^{-\lambda T c} E_{Q^\eta}[e^{\lambda T L_T^\pi}].$$

Using the definitions of $\underline{J}$ and $\Lambda(\lambda)$ we thus see that $\underline{J}(c) \leq -\lambda c + \Lambda(\lambda)$ for any $\lambda < 0$. This gives the upper bound $\underline{J}(c) \leq -\sup_{\lambda<0}\{\lambda c - \Lambda(\lambda)\} = -\underline{\Lambda}^*(c)$. $\square$

We are going to show that also the converse inequality holds, and this will establish

- the *duality relation* $\underline{J} = -\underline{\Lambda}^*$.

Moreover, we are going to identify

- an *optimal investment strategy* $\widehat{\pi}^c \in \mathcal{A}$
- and a *worst-case model* $Q^{\widehat{\eta}^c} \in \mathcal{Q}$

for the asymptotic minimization of robust downside risk (12), i. e.,

$$\underline{J}(c) = \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^{\widehat{\pi}^c} \leq c] = \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln Q^{\widehat{\eta}^c}[L_T^{\widehat{\pi}^c} \leq c] = \underline{J}_{Q^{\widehat{\eta}^c}}(c).$$

For this purpose, we proceed in three steps:

Step 1: The duality approach requires to compute the rates $\Lambda(\lambda)$, $\lambda < 0$. To this end, we use that

$$\Lambda(\lambda) = \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda] \tag{17}$$

can be interpreted as the optimal growth rate of robust expected power utility $u(x) = \frac{1}{\lambda}x^\lambda$ with risk aversion parameter $\lambda < 0$. More precisely, a priori estimates and case studies suggest that the distance between the maximal robust utility

$$U_T^\lambda(x_0) := \sup_{\pi\in\mathcal{A}_T} \inf_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[u(X_T^\pi)] = \tfrac{1}{\lambda} \inf_{\pi\in\mathcal{A}_T} \sup_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda] < 0 \tag{18}$$

and its upper bound 0 will decrease exponentially as $T \uparrow \infty$. Instead of analyzing the finite horizon problem (18) directly, it is natural to study its asymptotic version and to compute

- the *optimal rate of exponential decay* $\Lambda(\lambda)$ in (17),
- an *optimal long term investment strategy* $\pi^*(\lambda) \in \mathcal{A}$,
- and an *asymptotic worst-case model* $Q^{\eta^*(\lambda)} \in \mathcal{Q}$

for robust utility maximization. These asymptotic results are of intrinsic interest for utility maximizers with long but finite horizon, see, e. g., [2], [8], [10], [16] for a discussion in the non-robust case. Here they will be crucial to set up the duality approach. The analysis is carried out in Section 3.

Step 2: We study the non-robust version of (12) for specific measures $Q^\eta \in \mathcal{Q}$ and establish the duality relation

$$\underline{J}_{Q^\eta} = -\underline{\Lambda}^*_{Q^\eta}(c) = -\sup_{\lambda<0}\{\lambda c - \Lambda_{Q^\eta}(\lambda)\}.$$

In particular, this analysis extends the results in [18] to a stochastic factor model with nonlinear coefficients.

Step 3: Finally, we are going to identify a model $Q^{\widehat{\eta}^c} = Q^{\eta^*(\lambda[c])} \in \mathcal{Q}$ such that $\underline{\Lambda}^*(c) = \underline{\Lambda}^*_{Q^{\widehat{\eta}^c}}(c)$. *Step 2* then implies $\underline{J}(c) \geq \underline{J}_{Q^{\widehat{\eta}^c}}(c) = -\underline{\Lambda}^*_{Q^{\widehat{\eta}^c}}(c) = -\underline{\Lambda}^*(c)$. By Proposition 2.1 this yields the duality formula $\underline{J} = -\underline{\Lambda}^*$.

3 Optimal growth rates of robust expected power utility

We first analyze the asymptotics of robust expected power utility with parameter $\lambda < 0$. Conceptually the dual saddle point problem (17) would lead to a *stochastic differential game* on an infinite time horizon. Inspired by the papers [31], [20] on robust utility maxization problems with finite horizon we develop an alternative approach: The main idea consists in combining the *duality approach to robust utility maximization* (see, e. g., [30], [32], [12]) with methods from *risk-sensitive control* (see, e. g., [2], [3], [8], [9], [10], [25], [28]). Our main results will characterize the optimal growth rate $\Lambda(\lambda)$, an asymptotic worst-case model $Q^{\eta^*(\lambda)}$, and an optimal long term investment strategy $\pi^*(\lambda)$ in terms of an *ergodic Bellman equation.* As a byproduct, the duality approach also characterizes the exponential decay of the maximal robust utility $U_T^\lambda(x_0)$ as $T \uparrow \infty$. This method is also used in [24] to describe the asymptotics of robust power utility with parameter $\lambda \in (0,1)$. To give a self-contained presentation, we sketch the main ideas, but refer to [23] whenever the argument is essentially the same as in the case $\lambda \in (0,1)$.

3.1 Duality methods for robust utility maximization

In order to transform the primal saddle-point problem (17) into a simpler minimization problem on the dual side, we use the duality approach to robust utility maximization. Our exposition is based on Schied and Wu [32] for utility functions u on the positive halfline.

Let us denote by $v(y) := \sup_{x>0}\{u(x) - xy\}$, $y > 0$, the convex conjugate function of u, and consider the dual value function

$$V_T(y) := \inf_{Q\in\mathcal{Q}} \inf_{Y\in\mathcal{Y}^Q} E_Q[v(yY_T/S_T^0)], \quad y > 0, \tag{19}$$

for the finite horizon problem (18), defined in terms of the class of supermartingales

$$\mathcal{Y}_T^Q := \{Y \geq 0 | Y_0 = 1 \,\&\, \forall\, \pi \in \mathcal{A}_T : (Y_t X_t^\pi / S_t^0)_{t\leq T} \text{ is a } Q\text{-supermartingale}\}.$$

Note that $\mathcal{Y}_T^Q$ contains the density processes (taken with respect to Q and the numéraire S^0) of the class $\mathcal{P}_T$ of all equivalent local martingale measures on $(\Omega, \mathcal{F}_T)$. For a power utility function $u(x) = \lambda^{-1}x^\lambda$, the convex conjugate function takes the form $v(y) = -\beta^{-1}y^\beta$, $\beta := \frac{\lambda}{\lambda-1}$. We thus have the scaling property $V_T(y) = y^\beta V_T(1)$, and it follows from [32], Theorem 2.2, that the primal value function (18) is given by the duality formula

$$U_T^\lambda(x_0) = \inf_{y>0}\{V_T(y) + x_0 y\} = \tfrac{1}{\lambda} x_0^\lambda (-\beta V_T(1))^{1-\lambda}. \tag{20}$$

Moreover, [32], Theorem 2.6, shows that

$$V_T(1) = \inf_{Q\in\mathcal{Q}} \inf_{P\in\mathcal{P}_T} E_Q[v(\tfrac{dP}{dQ}|_{\mathcal{F}_T}/S_T^0)]. \tag{21}$$

In order to apply dynamic programming techniques to the dual problem, we parameterize the sets $\mathcal{Y}_T^Q$ and $\mathcal{P}_T$ by additional controls ν. First, it is easy to see that the density process of any martingale measure $P \in \mathcal{P}_T$ with respect to Q_0 takes the form

$$\tfrac{dP}{dQ_0}\big|_{\mathcal{F}_t} = Z_t^\nu := \mathcal{E}(-\int_0^\cdot \theta(Y_s)\, dW_s^1 - \int_0^\cdot \nu_s\, dW_s^2)_t \tag{22}$$

for some progressively measurable process ν such that $\int_0^T \nu_s^2\, ds < \infty$ Q_0-a. s.. Conversely, for any bounded process ν the Radon-Nikodým density Z_T^ν defines an equivalent local martingale measure on $(\Omega, \mathcal{F}_T)$. Up to any finite horizon T our market model thus admits a whole class of equivalent local martingale measures, and so its restriction to a finite horizon is *arbitrage-free* and *incomplete*. Second, if

$$\mathcal{M} := \{\nu = (\nu_t)_{t\geq 0} | \nu \text{ progressively measurable with } \int_0^T \nu_t^2\, dt < \infty\ Q_0\text{-a. s.}, \forall T > 0\},$$

then every $\nu \in \mathcal{M}$ defines a positive Q_0-supermartingale Z^ν via (22). Using Itô's formula one easily shows that $(D^\eta)^{-1} Z^\nu X^\pi / S^0$ is a positive local martingale under Q^η for any $\nu \in \mathcal{M}$ and $\pi \in \mathcal{A}_T$, and hence a Q^η-supermartingale. Combining these facts we obtain

$$\{(\tfrac{dP}{dQ^\eta}\big|_{\mathcal{F}_t})_{t\leq T} | P \in \mathcal{P}_T\} \subset \{((D_t^\eta)^{-1} Z_t^\nu)_{t\leq T} | \nu \in \mathcal{M}\} \subset \mathcal{Y}_T^{Q^\eta}.$$

In view of (19), (21) and (20) this inclusion and a change of measure finally yield the duality formula

$$U_T^\lambda(x_0) = \tfrac{1}{\lambda} x_0^\lambda (\sup_{\eta\in\mathcal{C}} \sup_{\nu\in\mathcal{M}} E_{Q_0}[(Z_T^\nu (S_T^0)^{-1})^{\frac{\lambda}{\lambda-1}} (D_T^\eta)^{\frac{1}{1-\lambda}}])^{1-\lambda} \tag{23}$$

whose right-hand side involves the maximization among the controls $\eta \in \mathcal{C}$ and $\nu \in \mathcal{M}$.

3.2 Dynamic programming methods

In a second step, we translate (23) into a standard control problem and derive *heuristically* an ergodic Bellman equation for the optimal growth rate $\Lambda(\lambda)$ by applying dynamic programming methods to the dual maximization problem. Since Z_T^ν, D_T^η, and the bond price S_T^0 depend on the factor process Y, the expectation at the right-hand side of (23) is a function of the initial state $Y_0 = y$, and we thus write

$$V(\lambda, \eta, \nu, y, T) := E_{Q_0}[(Z_T^\nu (S_T^0)^{-1})^{\frac{\lambda}{\lambda-1}} (D_T^\eta)^{\frac{1}{1-\lambda}}], \quad \eta \in \mathcal{C}, \nu \in \mathcal{M}. \tag{24}$$

Inserting the explicit representations of Z_T^ν, D_T^η and S_T^0, it follows that

$$V(\lambda, \eta, \nu, y, T) = E_{Q_0}[\mathcal{E}_T^{\eta,\nu} e^{\int_0^T l(\lambda, \eta_t, \nu_t, Y_t)\, dt}], \tag{25}$$

where the function $l : (-\infty, 0) \times \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_-$ is given by

$$l(\lambda, \eta, \nu, y) := \tfrac{1}{2} \tfrac{\lambda}{(1-\lambda)^2} [(\theta(y) + \eta^{11} y + \eta^{21})^2 + (\nu + \eta^{12} y + \eta^{22})^2] + \tfrac{\lambda}{1-\lambda} r(y) \tag{26}$$

and

$$\mathcal{E}_T^{\eta,\nu} := \mathcal{E}(\tfrac{1}{1-\lambda}(\int_0^\cdot \lambda\theta(Y_t) + \eta_t^{11} Y_t + \eta_t^{21}\, dW_t^1 + \int_0^\cdot \lambda\nu_t + \eta_t^{12} Y_t + \eta_t^{22}\, dW_t^2))_T.$$

To eliminate the Itô exponential $\mathcal{E}_T^{\eta,\nu}$ in the expression for $V(\lambda, \eta, \nu, y, T)$, it will be interpreted as the density of a new probability measure $R^{\eta,\nu}$ on $(\Omega, \mathcal{F}_T)$. Clearly, this requires the martingale condition $E_{Q_0}[\mathcal{E}_T^{\eta,\nu}] = 1$ which may be violated for arbitrary $\nu \in \mathcal{M}$. However, later on it will be enough to focus on nice controls ν such that this interpretation is indeed justified. Using this change of measure, we obtain

$$V(\lambda, \eta, \nu, y, T) = E_{R^{\eta,\nu}}[e^{\int_0^T l(\lambda, \eta_t, \nu_t, Y_t)\, dt}]. \tag{27}$$

By Girsanov's theorem, the dynamics of the factor process $(Y_t)_{t\le T}$ under $R^{\eta,\nu}$ is determined by the SDE

$$dY_t = h(\lambda, \eta_t, \nu_t, Y_t)\, dt + \rho\, dW_t^{\eta,\nu}. \tag{28}$$

Here $W^{\eta,\nu}$ is a two-dimensional $R^{\eta,\nu}$-Wiener process, and the function h is defined by

$$h(\lambda, \eta, \nu, y) := g(y) + \tfrac{\rho_1}{1-\lambda}(\lambda\theta(y) + \eta^{11} y + \eta^{21}) + \tfrac{\rho_2}{1-\lambda}(\lambda\nu + \eta^{12} y + \eta^{22}). \tag{29}$$

Combining (23), (24) and (27) now allows us to deduce that

$$U_T^\lambda(x_0) = \tfrac{1}{\lambda} x_0^\lambda v(y, T)^{1-\lambda}, \tag{30}$$

where

$$v(y, T) := \sup_{\eta\in\mathcal{C}} \sup_{\nu\in\mathcal{M}} E_{R^{\eta,\nu}}[e^{\int_0^T l(\lambda,\eta_t,\nu_t,Y_t)\, dt}]$$

denotes the value function of the finite horizon optimization problem on the dual side of (23). Such an "expected exponential of integral cost criterion" with a dynamics of the form (28) is standard in stochastic control theory; see, e. g., [11], Remark IV.3.3. As a result, the value function v can be characterized as the solution to the *Hamilton-Jacobi-Bellman* (HJB) equation

$$v_t = \tfrac{1}{2}\|\rho\|^2 v_{yy} + \sup_{\eta\in\Gamma}\sup_{\nu\in\mathbb{R}}\{l(\lambda, \eta, \nu, \cdot)v + h(\lambda, \eta, \nu, \cdot)v_y\}, \quad v(\cdot, 0) \equiv 1. \tag{31}$$

The discussion in *Step 1* (cf. p. 6) and (30) now suggest that the optimal growth rate $\Lambda(\lambda)$ in (17) satisfies

$$\Lambda(\lambda) = \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln |U_T^\lambda(x_0)| = \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln(v(y, T)^{1-\lambda}).$$

This motivates the heuristic Ansatz (cf. Fleming and McEneaney [7])

$$(1-\lambda) \ln v(y, T) = \ln |U_T^\lambda(x_0)| \approx \Lambda(\lambda)T + \varphi(\lambda, y) \tag{32}$$

i. e., a formal separation of time and space variables. Using this separation to compute the partial derivatives in (31), we obtain an *ergodic Bellman equation* (EBE) (see, e. g., [1], [22], [27], and the references therein) for the pair $(\Lambda(\lambda), \varphi(\lambda, \cdot))$:

$$\Lambda(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \tfrac{1}{1-\lambda}\varphi_y^2] + \sup_{\eta\in\Gamma}\sup_{\nu\in\mathbb{R}}\{(1-\lambda)l(\lambda, \eta, \nu, \cdot) + \varphi_y h(\lambda, \eta, \nu, \cdot)\}. \tag{33}$$

In the next step, we are going to identify heuristically a candidate for the optimal long term investment process $\pi^*(\lambda)$. To this end, let us assume that the EBE (33) has a solution $\Lambda(\lambda) \in \mathbb{R}_-$, $\varphi(\lambda, \cdot) \in C^2(\mathbb{R})$. Since the maximizer $\nu^*(\lambda, \eta, y)$ among all $\nu \in \mathbb{R}$ can be computed explicitly as

$$\nu^*(\lambda, \eta, y) = -\eta^{12} y - \eta^{22} - \varphi_y(\lambda, y)\rho_2, \tag{34}$$

the EBE (33) actually involves only a supremum among the set Γ. Suppose that $\eta^*(\lambda, y)$ is a maximizer in (33), let $Q^{\eta^*(\lambda)} \in \mathcal{Q}$ be the probabilistic model corresponding to the feedback control $\eta_t^*(\lambda) = \eta^*(\lambda, Y_t)$, and suppose that $Q^{\eta^*(\lambda)} \in \mathcal{Q}$ is a worst-case model in the asymptotic sense that

$$\Lambda(\lambda) = \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln |U_T^\lambda(x_0)| = \varlimsup_{T\uparrow\infty} \tfrac{1}{T} \ln \inf_{\pi\in\mathcal{A}_T} E_{Q^{\eta^*(\lambda)}}[(X_T^\pi)^\lambda]. \tag{35}$$

Later on we will show that this assumption is indeed justified. We are now going to introduce a change of measure which will allow us to interpret the finite time minimization problem at the right-hand side of (35) as an exponential of integral criterion. To this end, note that for any $\pi \in \mathcal{A}$ the unique strong solution to (9) takes the form

$$X_T^\pi = x_0 e^{\int_0^T \pi_t\sigma\, dW_t^{1,\eta^*(\lambda)} + \int_0^T r(Y_t)+\pi_t\sigma(\theta(Y_t)+\eta_t^{11,*}(\lambda)Y_t+\eta_t^{21,*}(\lambda))-\frac{1}{2}\pi_t^2\sigma^2\, dt}.$$

In terms of the function

$$\widetilde{l}(\lambda,\pi,\eta,y) := \tfrac{1}{2}\lambda(\lambda-1)\sigma^2\pi^2 + \lambda\sigma[\theta(y)+\eta^{11}y+\eta^{21}]\pi + \lambda r(y) \tag{36}$$

and of the probability measure R^π on $(\Omega, \mathcal{F}_T)$ defined by the Radon-Nikodým density

$$\tfrac{dR^\pi}{dQ^{\eta^*(\lambda)}}\Big|_{\mathcal{F}_T} := \mathcal{E}\Big(\int_0^\cdot \lambda\pi_t\sigma\, dW_t^{1,\eta^*}\Big)_T,$$

the expectation at the right-hand side of (35) can thus be rewritten as

$$E_{Q^{\eta^*(\lambda)}}[(X_T^\pi)^\lambda] = x_0^\lambda E_{R^\pi}[e^{\int_0^T \widetilde{l}(\lambda,\pi_t,\eta^*(\lambda,Y_t),Y_t)\, dt}].$$

By Girsanov's theorem, the factor process $(Y_t)_{t\le T}$ evolves under R^π according to

$$dY_t = \widetilde{h}(\lambda,\pi_t,\eta_t^*(\lambda),Y_t)\, dt + \rho\, dW_t^\pi, \tag{37}$$

where W^π is a one-dimensional Wiener process W^π, and the function $\widetilde{h}$ is defined by

$$\widetilde{h}(\lambda,\pi,\eta,y) := g(y) + (\rho, \eta^{1\cdot}y+\eta^{2\cdot}) + \lambda\rho_1\sigma\pi. \tag{38}$$

We thus see that the finite horizon maximization problem appearing in the right-hand side of (35) can be viewed as a finite horizon control problem with value function

$$\widetilde{v}(y,T) := \inf_{\pi\in\mathcal{A}_T} E_{Q^{\eta^*(\lambda)}}[(X_T^\pi)^\lambda] = x_0^\lambda \inf_{\pi\in\mathcal{A}_T} E_{R^\pi}[e^{\int_0^T \widetilde{l}(\lambda,\pi_t,\eta^*(\lambda,Y_t),Y_t)\, dt}]$$

and with dynamics (37). In analogy to (31) $\widetilde{v}$ is expected to satisfy the HJB equation

$$\widetilde{v}_t = \tfrac{1}{2}\|\rho\|^2\widetilde{v}_{yy} + \inf_{\pi\in\mathbb{R}}\{\widetilde{l}(\lambda,\pi,\eta^*,\cdot)\widetilde{v} + \widetilde{h}(\lambda,\pi,\eta^*,\cdot)\widetilde{v}_y\}, \quad \widetilde{v}(\cdot,0)\equiv x_0^\lambda. \tag{39}$$

Our Ansatz (32) combined with (35) for the worst-case measure $Q^{\eta^*(\lambda)}$ now motivates the heuristic Ansatz $\ln\widetilde{v}(y,T)\approx\Lambda(\lambda)T+\varphi(\lambda,y)$. Inserting this asymptotic identity into (39), we derive an alternative version of the EBE:

$$\Lambda(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy}(\lambda,\cdot)+\varphi_y^2(\lambda,\cdot)] + \inf_{\pi\in\mathbb{R}}\{\widetilde{l}(\lambda,\pi,\eta^*,\cdot)+\varphi_y(\lambda,\cdot)\widetilde{h}(\lambda,\pi,\eta^*,\cdot)\}. \tag{40}$$

Note that the role played by the controls η and ν in (33) is now taken over by the "trading strategies" π. We expect that the minimizing function

$$\pi^*(\lambda,y) = \tfrac{1}{1-\lambda}\tfrac{1}{\sigma}(\varphi_y(\lambda,y)\rho_1+\theta(y)+\eta^{11,*}(\lambda,y)y+\eta^{12,*}(\lambda,y)). \tag{41}$$

in (40) provides an optimal feedback control $\pi_t^*(\lambda)=\pi^*(\lambda,Y_t)$, $t\ge 0$, for the asymptotic maximization of power utility with respect to the specific model $Q^{\eta^*(\lambda)}$ and at the same time for the original robust problem (17).

3.3 Verification theorems

Let us first show that the value $\tilde{\Lambda}(\lambda)$ given by a specific solution to the EBE (33) decribes indeed the exponential decay of the maximal robust utility $|U_T^\lambda(x_0)|$ as T tends to infinity. For this purpose, we introduce

Assumption 3.1. *There exists a solution $\tilde{\Lambda}(\lambda) \in \mathbb{R}_-$, $\varphi(\lambda, \cdot) \in C^2(\mathbb{R})$ to the EBE*

$$\tilde{\Lambda}(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \tfrac{1}{1-\lambda}\varphi_y^2] + \sup_{\eta\in\Gamma}\sup_{\nu\in\mathbb{R}}\{(1-\lambda)l(\lambda,\eta,\nu,\cdot) + \varphi_y h(\lambda,\eta,\nu,\cdot)\} \tag{42}$$

which fulfills the following regularity conditions:

a) $|\varphi_y(\lambda, y)| \leq C_1(\lambda)(1 + |y|)$

b) *For the function κ defined by*

$$\kappa(\lambda,\eta,y) := g(y) + \tfrac{\lambda}{1-\lambda}\rho_1(\theta(y) + \eta^{11}y + \eta^{21}) + (\rho, \eta^{1\cdot}y + \eta^{2\cdot}) + [\tfrac{1}{1-\lambda}\rho_1^2 + \rho_2^2]\varphi_y(\lambda, y)$$

there exist constants $C_2(\lambda), C_3(\lambda) > 0$ such that

$$y\kappa(\lambda,\eta,y) \leq -C_2(\lambda)y^2 + C_3(\lambda) \quad \text{for all } \eta \in \Gamma. \tag{43}$$

c) *Let $\widehat{R}^\eta$ be the probability measure such that Y evolves according to the SDE*

$$dY_t = \kappa(\lambda, \eta_t, Y_t)\,dt + \rho\,d\widehat{W}_t^\eta, \tag{44}$$

where $\widehat{W}^\eta$ is a two-dimensional Wiener process under $\widehat{R}^\eta$. Then

$$\lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{\eta\in\mathcal{C}} E_{\widehat{R}^\eta}[e^{-\varphi(\lambda, Y_T)}] = 0. \tag{45}$$

The main challenge is to derive the existence of such a *regular* solution. Existence results for EBEs (see, e. g., [1], [27], [22], and the references therein) do not cover (42) in its general form. On the other hand, there may exist multiple solutions $(\tilde{\Lambda}(\lambda), \varphi(\lambda, \cdot))$ (see, e. g., Subsection 5.2), even beyond the fact that φ is determined only except for an additive constant. However, the verification theorems will require a "uniform ergodicity condition" such as c), and this condition selects the "good candidate" for the optimal growth rate $\Lambda(\lambda)$. In this paper we do not try to study the existence problem rigorously. Instead we will discuss in Section 5 three examples which allow for a regular solution in the sense of Assumption 3.1.

Theorem 3.1. *Under Assumption 3.1 it follows that*

$$\tilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln(\sup_{\eta\in\mathcal{C}}\sup_{\nu\in\mathcal{M}} V(\lambda,\eta,\nu,y_0,T)^{1-\lambda}) \quad \text{for any } Y_0 = y_0. \tag{46}$$

The suprema are attained for controls $\eta_t^*(\lambda) := \eta^*(\lambda, Y_t)$, $\nu_t^*(\lambda) := \nu^*(\lambda, Y_t)$, $t \geq 0$, defined in terms of a measurable Γ-valued function $\eta^*(\lambda, \cdot)$ and the function

$$\nu^*(\lambda, y) := -\eta^{12,*}(\lambda, y)y - \eta^{22,*}(\lambda, y) - \varphi_y(\lambda, y)\rho_2$$

which realize the suprema in the EBE (42). This means that

$$\tilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln V(\lambda, \eta^*(\lambda), \nu^*(\lambda), y_0, T)^{1-\lambda}. \tag{47}$$

In particular, the duality methods for robust utility maximization imply that

$$\tilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln |U_T^\lambda(x_0)| = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln |U_T^{\lambda, Q^{\eta^*(\lambda)}}(x_0)| \quad \text{for any } X_0^\pi = x_0. \tag{48}$$

Proof. 1) To keep the notation as simple as possible, the dependence on the fixed parameter $\lambda < 0$ will be mostly suppressed throughout this proof. We are first going to show that the value $\tilde{\Lambda}(\lambda)$ provides an upper bound for the rate of exponential decay of $|U_T^\lambda(x_0)|$. To this end, we use the duality relation (cf. (23))

$$U_T^\lambda(x_0) = \tfrac{1}{\lambda} x_0^\lambda \sup_{\eta \in \mathcal{C}} \sup_{\nu \in \mathcal{M}} V(\eta, \nu, y, T)^{1-\lambda}, \tag{49}$$

derive suitable upper bounds for any fixed horizon T, and then pass to the limit. Moreover, [20], Lemma 3.2, guarantees that, for all controls $\eta \in \mathcal{C}$,

$$\sup_{\nu \in \mathcal{M}} V(\eta, \nu, y, T) = \sup_{\nu \in \mathcal{M}^0} V(\eta, \nu, y, T), \tag{50}$$

where $\mathcal{M}^0 := \{\nu \in \mathcal{M} | \forall T > 0 : \int_0^T \nu_t^2 \, dt \text{ is } Q_0\text{-a.s. bounded}\}$. This allows us to concentrate on a subset of nice controls ν for which all subsequent Girsanov transformations are valid. Let us fix controls $\eta \in \mathcal{C}$, $\nu \in \mathcal{M}^0$, and let T be a given maturity. Due to (27), we have

$$V(\eta, \nu, y_0, T) = E_{R^{\eta,\nu}}[e^{\int_0^T l(\eta_t, \nu_t, Y_t)\, dt}],$$

where l is given by (26), and where the probability measure $R^{\eta,\nu}$ is defined on $(\Omega, \mathcal{F}_T)$ by the Itô exponential $\mathcal{E}_T^{\eta,\nu}$; cf. p. 8. By Girsanov's theorem, the dynamics of the factor process Y under $R^{\eta,\nu}$ takes the form

$$dY_t = h(\eta_t, \nu_t, Y_t)\, dt + \rho\, dW_t^{\eta,\nu}$$

for the function h defined by (29) and for a two-dimensional $R^{\eta,\nu}$-Wiener process $W^{\eta,\nu}$. To eliminate later on the control ν in the dynamics of Y, we define

$$\gamma(\eta, \nu, y) := (1-\lambda) l(\eta, \nu, y) + \varphi_y(y) h(\eta, \nu, y) - \sup_{\nu \in \mathbb{R}} \{(1-\lambda) l(\eta, \nu, y) + \varphi_y(y) h(\eta, \nu, y)\}.$$

This auxiliary function $\gamma \leq 0$ can be rewritten in the condensed form

$$\gamma(\eta, \nu, y) = \tfrac{1}{2} \tfrac{\lambda}{1-\lambda} (\nu - \nu^*(\eta, y))^2 \tag{51}$$

by inserting the maximizer $\nu^*(\eta, y)$ in (34). In terms of γ the EBE (42) yields

$$\tilde{\Lambda}(\lambda) \geq \tfrac{1}{2} \|\rho\|^2 [\varphi_{yy} + \tfrac{1}{1-\lambda} \varphi_y^2] + (1-\lambda) l(\eta, \nu, \cdot) + \varphi_y h(\eta, \nu, \cdot) - \gamma(\eta, \nu, \cdot). \tag{52}$$

Itô's formula applied to $\varphi(\lambda, \cdot) \in C^2(\mathbb{R})$ combined with this inequality then implies

$$\begin{aligned} \varphi(Y_T) - \varphi(y_0) &= \int_0^T (\varphi_y(Y_t) h(\eta_t, \nu_t, Y_t) + \tfrac{1}{2} \|\rho\|^2 \varphi_{yy}(Y_t))\, dt + \int_0^T \varphi_y(Y_t) \rho\, dW_t^{\eta,\nu} \\ &\leq \int_0^T (\tilde{\Lambda}(\lambda) - \tfrac{1}{2} \tfrac{1}{1-\lambda} \|\rho\|^2 \varphi_y^2(Y_t) + \gamma(\eta_t, \nu_t, Y_t) - (1-\lambda) l(\eta_t, \nu_t, Y_t))\, dt \\ &\qquad + \int_0^T \varphi_y(Y_t) \rho\, dW_t^{\eta,\nu}. \end{aligned}$$

Dividing through $1 - \lambda$, rearranging the terms, and taking the exponential on both sides, we thus derive the inequality

$$\begin{aligned} V(\eta, \nu, y_0, T) =& E_{R^{\eta,\nu}}[e^{\int_0^T l(\lambda, \eta_t, \nu_t, Y_t)\, dt}] \\ \leq& E_{R^{\eta,\nu}}[e^{\frac{1}{1-\lambda}(\tilde{\Lambda}(\lambda) T + \varphi(y_0) - \varphi(Y_T) + \int_0^T \gamma(\eta_t, \nu_t, Y_t)\, dt)} \mathcal{E}(\int_0^\cdot \tfrac{\varphi_y(Y_t)}{1-\lambda} \rho\, dW_t^{\eta,\nu})_T]. \end{aligned} \tag{53}$$

To eliminate the Itô exponential $\mathcal{E}(\cdot)_T$, we use the change of measure

$$\tfrac{d\overline{R}^{\eta,\nu}}{dR^{\eta,\nu}}\big|_{\mathcal{F}_T} := \mathcal{E}\Big(\int_0^{\cdot} \tfrac{\varphi_y(Y_t)}{1-\lambda}\rho\, dW_t^{\eta,\nu}\Big)_T.$$

By Girsanov's theorem, the dynamics of Y under $\overline{R}^{\eta,\nu}$ is governed by the SDE

$$dY_t = [h(\eta_t,\nu_t,Y_t) + \tfrac{1}{1-\lambda}\|\rho\|^2\varphi_y(Y_t)]\,dt + \rho\, d\overline{W}_t^{\eta,\nu},$$

where $\overline{W}^{\eta,\nu}$ is a two-dimensional Wiener process under $\overline{R}^{\eta,\nu}$. But this dynamics still depends on the irrepressible control ν. To remove this dependence, we define the probability measure $\widehat{R}^\eta$ on $(\Omega,\mathcal{F}_T)$ in terms of the Radon-Nikodým density

$$\tfrac{d\widehat{R}^{\eta}}{d\overline{R}^{\eta,\nu}}\big|_{\mathcal{F}_T} := \mathcal{E}\Big(\int_0^{\cdot} \tfrac{\lambda}{1-\lambda}(\nu^*(\eta_t,Y_t)-\nu_t)\, d\overline{W}_t^{2,\eta,\nu}\Big)_T.$$

Using again Girsanov's theorem, we see that Y evolves under $\widehat{R}^\eta$ according to

$$\begin{aligned} dY_t &= [h(\eta_t,\nu^*(\eta_t,Y_t),Y_t) + \tfrac{1}{1-\lambda}\|\rho\|^2\varphi_y(Y_t)]\,dt + \rho\, d\widehat{W}_t^{\eta} \\ &= \kappa(\eta_t,Y_t)\,dt + \rho\, d\widehat{W}_t^{\eta}. \end{aligned} \tag{54}$$

Here $\widehat{W}^\eta$ is a two-dimensional $\widehat{R}^\eta$-Wiener process, and the drift function κ defined in Assumption 3.1 b) does no longer depend on ν. Together these measure transformations translate (53) into the estimate

$$\begin{aligned} V(\eta,\nu,y_0,T) &\le E_{\overline{R}^{\eta,\nu}}\big[e^{\frac{1}{1-\lambda}(\tilde\Lambda(\lambda)T+\varphi(y_0)-\varphi(Y_T)+\int_0^T\gamma(\eta_t,\nu_t,Y_t)\,dt)}\big] \\ &= E_{\widehat{R}^\eta}\big[e^{\frac{1}{1-\lambda}(\tilde\Lambda(\lambda)T+\varphi(y))-\varphi(Y_T)+\int_0^T\gamma(\eta_t,\nu_t,Y_t)\,dt)}\tfrac{d\overline{R}^{\eta,\nu}}{d\widehat{R}^\eta}\big|_{\mathcal{F}_T}\big]. \end{aligned} \tag{55}$$

To eliminate the density $d\overline{R}^{\eta,\nu}/d\widehat{R}^\eta|_{\mathcal{F}_T}$, we define $p := \frac{\lambda-1}{\lambda} > 1$ and apply Hölder's inequality with $1/p+1/q=1$ to the right-hand side of (55). This yields

$$V(\eta,\nu,y_0,T) \le E_{\widehat{R}^\eta}\big[e^{\frac{q}{1-\lambda}(\tilde\Lambda(\lambda)T+\varphi(y_0)-\varphi(Y_T))}\big]^{1/q}E_{\widehat{R}^\eta}\big[\big(\tfrac{d\overline{R}^{(\eta,\nu)}}{d\widehat{R}^{(\eta)}}\big|_{\mathcal{F}_T}e^{\frac{1}{1-\lambda}\int_0^T\gamma(\eta_t,\nu_t,Y_t)\,dt}\big)^p\big]^{1/p}.$$

But in view of (51) and our choice of p we see that

$$\Big(\tfrac{d\overline{R}^{\eta,\nu}}{d\widehat{R}^\eta}\big|_{\mathcal{F}_T}e^{\frac{1}{1-\lambda}\int_0^T\gamma(\eta_t,\nu_t,Y_t)\,dt}\Big)^p = \mathcal{E}\Big(\int_0^{\cdot}\tfrac{p\lambda}{1-\lambda}(\nu^*(\eta_t,Y_t)-\nu_t)\,d\widehat{W}_t^{2,\eta}\Big)_T.$$

Since the Itô exponential is a martingale up to time T, it follows that

$$V(\eta,\nu,y_0,T) \le e^{\frac{1}{1-\lambda}(\tilde\Lambda(\lambda)T+\varphi(y_0))}E_{\widehat{R}^\eta}\big[e^{-\varphi(Y_T)}\big]^{1/1-\lambda} \quad \text{for all } \eta\in\mathcal{C},\nu\in\mathcal{M}^0.$$

Recall now that the factor process Y admits under $\widehat{R}^\eta$ the dynamics (44). Thus our Assumption 3.1 c) ensures that

$$\overline{\lim_{T\uparrow\infty}}\ \tfrac{1}{T}\ln\big(\sup_{\eta\in\mathcal{C}}\sup_{\nu\in\mathcal{M}^0}V(\eta,\nu,y_0,T)^{1-\lambda}\big) \le \tilde\Lambda(\lambda).$$

Taking also into account (50), this finally yields the upper bound in (46)

$$\overline{\lim_{T\uparrow\infty}}\ \tfrac{1}{T}\ln\big(\sup_{\eta\in\mathcal{C}}\sup_{\nu\in\mathcal{M}}V(\eta,\nu,y_0,T)^{1-\lambda}\big) \le \tilde\Lambda(\lambda). \tag{56}$$

2) To establish equality in (56) we identify controls $\eta^* \in \mathcal{C}$ and $\nu^* \in \mathcal{M}$ such that

$$\tilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln V(\eta^*, \nu^*, y, T)^{1-\lambda}. \tag{57}$$

Since l, h and the maximizing function $\nu^*(\cdot, y)$ (cf. (34)) are continuous with respect to η and $\Gamma \subset \mathbb{R}^4$ is compact, there exists

$$\eta^*(y) \in \arg\max_{\eta\in\Gamma}\{(1-\lambda)l(\eta, \nu^*(\eta, y), y) + \varphi_y(y)h(\eta, \nu^*(\eta, y), y)\}$$

which by a measurable selection argument can be chosen as a measurable function η^* on $\mathbb{R}$. Let us write $\nu^*(y) := \nu^*(\eta^*(y), y)$, and let η^*, ν^* be the feedback controls defined by $\eta^*_t := \eta^*(Y_t)$, $\nu^*_t := \nu^*(Y_t)$, $t \geq 0$. Then η^* belongs to $\mathcal{C}$, and it is easy to show that $\nu^* \in \mathcal{M}$. Moreover, inserting $\eta^*(y)$, $\nu^*(y)$ we obtain $\gamma(\eta^*(y), \nu^*(y), y) = 0$, equality holds in (52), and the probability measure $\overline{R}^{\eta^*,\nu^*}$ coincides with $\widehat{R}^{\eta^*}$. Proceeding as in part 1) (cf. (55)), it thus follows that

$$V(\eta^*, \nu^*, y_0, T) = e^{\frac{1}{1-\lambda}(\tilde{\Lambda}(\lambda)T+\varphi(y_0))} E_{\widehat{R}^{\eta^*}}[e^{-\frac{1}{1-\lambda}\varphi(Y_T)}], \tag{58}$$

where Y evolves under $\widehat{R}^{\eta^*}$, as in (54), according to the SDE

$$dY_t = \kappa(\eta^*_t, Y_t)\, dt + \rho\, d\widehat{W}^{\eta^*}_t. \tag{59}$$

Since θ, φ_y, and $\nu^*(\eta^*(\cdot), \cdot)$ grow at most linearly, the measure transformations in part 1) can all be justified by [26], Example 3, Subsection 6.2.

Moreover, the linear growth Assumption 3.1 a) for φ_y ensures that $|\varphi(y)| \leq K(1 + y^2)$. By Jensen's inequality applied to (58) we thus see that

$$\begin{aligned}\varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln(V(\eta^*, \nu^*, y_0, T)^{1-\lambda}) &\geq \tilde{\Lambda}(\lambda) - \varlimsup_{T\uparrow\infty} \tfrac{1}{T} E_{\widehat{R}^{\eta^*}}[|\varphi(Y_T)|] \\ &\geq \tilde{\Lambda}(\lambda) - K \varlimsup_{T\uparrow\infty} \tfrac{1}{T} E_{\widehat{R}^{\eta^*}}[Y_T^2] = \tilde{\Lambda}(\lambda).\end{aligned}$$

Here the last equation follows from Assumption 3.1 b) and [24], Lemma 8.2 i), applied for the SDE (59). In view of (56), this implies (57), and so (46) follows.

3) It remains to translate these results to the initial problem of robust utility maximization: The finite horizon duality relation (20) holds for any (regular) convex class of measures, and in particular for the one-point set $\{Q^{\eta^*}\}$. In analogy to (23) it thus follows that the maximal value for expected power utility in the specific model Q^{η^*} satisfies the duality formula

$$U_T^{Q^{\eta^*}}(x_0) = \tfrac{1}{\lambda} x_0^\lambda \sup_{\nu\in\mathcal{M}} V(\eta^*, \nu, y_0, T)^{1-\lambda}.$$

Taking also into account the duality relation (49) for the whole set $\mathcal{Q}$, we derive (48) from (46) and (47). □

Remark 3.1. *If $\rho_2 \equiv 0$, then the process Y evolves under $\overline{R}^{\eta,\nu}$ according to*

$$dY_t = \kappa(\lambda, \eta_t, Y_t)\, dt + \rho_1\, d\overline{W}^{1,\eta,\nu}_t$$

for κ introduced in Assumption 3.1 b). This dynamics does no longer depend on the irrepressible control ν. Moreover, it follows from (55) and $\gamma(\eta, \nu, y) \leq 0$ that

$$V(\lambda, \eta, \nu, y_0, T) \leq e^{\frac{1}{1-\lambda}(\tilde{\Lambda}(\lambda)T+\varphi(\lambda,y_0))} E_{\overline{R}^{\eta,\nu}}[e^{-\frac{1}{1-\lambda}\varphi(\lambda,Y_T)}].$$

Thus (45) in Assumption 3.1 c) can be replaced by the weaker condition

$$\lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{\eta\in\mathcal{C}} E_{\overline{R}^{\eta,\nu}}[e^{-\frac{1}{1-\lambda}\varphi(\lambda,Y_T)}] = 0. \tag{60}$$

This observation will be crucial to identify the optimal growth rate $\Lambda(\lambda)$ in the geometric OU model with uncertain mean reversion; see Subsection 5.2.

To identify a long term trading strategy $\pi^*(\lambda)$ we need

Assumption 3.2. *Consider the solution $(\tilde{\Lambda}(\lambda), \varphi(\lambda,\cdot))$ to the EBE (42) described in Assumption 3.1, and let $\eta^*(\lambda,\cdot)$ be the corresponding maximizing function. Let $\widehat{Q}^\eta$ be the probability measure such that Y evolves according to the SDE*

$$dY_t = \widetilde{\kappa}(\lambda, \eta_t, Y_t)\, dt + \rho\, d\widehat{W}_t^\eta,$$

where $\widehat{W}$ denotes a two-dimensional $\widehat{Q}^\eta$-Wiener process, and where $\widetilde{\kappa}$ is given by

$$\begin{aligned}\widetilde{\kappa}(\lambda,\eta,y) := g(y) + \tfrac{\lambda}{1-\lambda}\rho_1(\theta(y) + \eta^{11,*}(\lambda,y)y + \eta^{21,*}(\lambda,y)) \\ + (\rho, \eta^{1\cdot}y + \eta^{2\cdot}) + [\tfrac{1}{1-\lambda}\rho_1^2 + \rho_2^2]\varphi_y(\lambda,y).\end{aligned}$$

Then

$$\lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{\eta\in\mathcal{C}} E_{\widehat{Q}^\eta}[e^{-\varphi(\lambda,Y_T)}] = 0.$$

Theorem 3.2. *If our regularity Assumptions 3.1, 3.2 are satisfied, then we have:*

i) *The value $\tilde{\Lambda}(\lambda)$ given by the solution to the ergodic Bellman equation (42) coincides with the optimal rate of exponential decay*

$$\Lambda(\lambda) = \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda]$$

of the distance between robust power utility and its upper bound zero. In view of (48) this particularly means that

$$\Lambda(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln |U_T^\lambda(x_0)| = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln |U_T^{\lambda,Q^{\eta^*(\lambda)}}(x_0)|,$$

where $Q^{\eta^*(\lambda)} \in \mathcal{Q}$ is the measure specified by the feedback control $\eta^*(\lambda)$.

ii) *In the specific model $Q^{\eta^*(\lambda)}$ the optimal rate of exponential decay of the distance between power utility and its upper bound zero is equal to $\Lambda(\lambda)$, i. e.,*

$$\Lambda(\lambda) = \Lambda_{Q^{\eta^*(\lambda)}}(\lambda) := \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln E_{Q^{\eta^*(\lambda)}}[(X_T^\pi)^\lambda].$$

iii) *The trading strategy $\pi_t^*(\lambda) = \pi^*(\lambda, Y_t)$, $t \geq 0$, defined in terms of the function*

$$\pi^*(\lambda,y) = \tfrac{1}{1-\lambda}\tfrac{1}{\sigma}(\varphi_y(\lambda,y)\rho_1 + \theta(y) + \eta^{11,*}(\lambda,y)y + \eta^{21,*}(\lambda,y)) \tag{61}$$

belongs to $\mathcal{A}$, and it is optimal in the sense that

$$\Lambda(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[(X_T^{\pi^*(\lambda)})^\lambda] = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln E_{Q^{\eta^*(\lambda)}}[(X_T^{\pi^*(\lambda)})^\lambda]. \tag{62}$$

Proof. In Theorem 3.1 we have seen that in the specific model $Q^{\eta^*(\lambda)}$ the distance between the maximal expected utility and its upper bound 0 decays exponentially with rate $\tilde{\Lambda}(\lambda)$, i. e.,

$$\tilde{\Lambda}(\lambda) = \lim_{T\uparrow\infty} \frac{1}{T} \ln |U_T^{\lambda, Q^{\eta^*(\lambda)}}(x_0)| = \lim_{T\uparrow\infty} \frac{1}{T} \ln \inf_{\pi\in\mathcal{A}_T} E_{Q^{\eta^*(\lambda)}}[(X_T^\pi)^\lambda]$$

for any initial capital $X_0^\pi = x_0$. Thus the inclusion $\mathcal{A} \subseteq \mathcal{A}_T$ yields the inequality

$$\tilde{\Lambda}(\lambda) \le \inf_{\pi\in\mathcal{A}} \underline{\lim}_{T\uparrow\infty} \frac{1}{T} \ln E_{Q^{\eta^*(\lambda)}}[(X_T^\pi)^\lambda] \le \inf_{\pi\in\mathcal{A}} \underline{\lim}_{T\uparrow\infty} \frac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[(X_T^\pi)^\lambda] = \Lambda(\lambda).$$

To show that the preceding inequalities are in fact equalities, it is enough to verify that the strategy $\pi^*(\lambda)$ belongs to $\mathcal{A}$ and to establish the estimate

$$\widetilde{\Lambda}(\lambda) \ge \overline{\lim}_{T\uparrow\infty} \frac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} E_{Q^\eta}[(X_T^{\pi^*(\lambda)})^\lambda]. \tag{63}$$

This yields the identity $\tilde{\Lambda}(\lambda) = \Lambda(\lambda) = \Lambda_{Q^{\eta^*(\lambda)}}(\lambda)$ and at the same time (62). The proof of (63) can be obtained in analogy to [24], Theorem 4.2. $\square$

4 A duality approach to the robust large deviations criterion

The following proposition will allow us to apply convex duality methods in order to compute the Fenchel-Legendre transforms $\underline{\Lambda}_Q^*$, $\underline{\Lambda}^*$.

Proposition 4.1. *For any $Q \in \mathcal{Q}$, the function*

$$\Lambda_Q : (-\infty, 0) \to \mathbb{R}_-, \quad \lambda \mapsto \inf_{\pi\in\mathcal{A}} \underline{\lim}_{T\uparrow\infty} \frac{1}{T} \ln E_Q[(X_T^\pi)^\lambda] \tag{64}$$

is convex. Moreover, convexity holds for the robust growth rate (15), viewed as a function Λ on the interval $(-\infty, 0)$.

Proof. To verify convexity for Λ_Q, take $\alpha \in (0,1)$ and $\lambda_1, \lambda_2 < 0$. For arbitrary $\epsilon > 0$ we choose trading strategies $\pi_1, \pi_2 \in \mathcal{A}$ such that

$$\Lambda_Q(\lambda_i) + \epsilon \ge \overline{\lim}_{T\uparrow\infty} \frac{1}{T} \ln E_Q[(X_T^{\pi_i})^{\lambda_i}], \quad i = 1, 2.$$

Furthermore, we define $\gamma := \alpha\lambda_1/(\alpha\lambda_1 + (1-\alpha)\lambda_2) \in (0,1)$ and $\widetilde{\pi} := \gamma\pi_1 + (1-\gamma)\pi_2$. Then $\widetilde{\pi}$ belongs to $\mathcal{A}$, and it satisfies $\widetilde{\pi}_t^2 \le \gamma\pi_{1t}^2 + (1-\gamma)\pi_{2t}^2$. Since $Q = Q^\eta$ for some process $\eta \in \mathcal{C}$, we now infer from the SDE (9) that

$$X_T^{\widetilde{\pi}} = (X_T^{\pi_1})^\gamma (X_T^{\pi_2})^{1-\gamma} e^{\frac{1}{2}\int_0^T \sigma^2[\gamma\pi_{1t}^2 + (1-\gamma)\pi_{2t}^2 - \widetilde{\pi}_t^2]\,dt} \ge (X_T^{\pi_1})^\gamma (X_T^{\pi_2})^{1-\gamma},$$

i. e.,

$$X_T^{\widetilde{\pi}} \ge ((X_T^{\pi_1})^{\alpha\lambda_1} (X_T^{\pi_2})^{(1-\alpha)\lambda_2})^{\frac{1}{\alpha\lambda_1 + (1-\alpha)\lambda_2}}.$$

Raised to the power of $\alpha\lambda_1 + (1-\alpha)\lambda_2 < 0$, the inequality is reversed. Hence,

$$\begin{aligned} E_Q[(X_T^{\widetilde{\pi}})^{\alpha\lambda_1 + (1-\alpha)\lambda_2}] &\le E_Q[(X_T^{\pi_1})^{\alpha\lambda_1} (X_T^{\pi_2})^{(1-\alpha)\lambda_2}] \\ &\le E_Q[(X_T^{\pi_1})^{\lambda_1}]^\alpha E_Q[(X_T^{\pi_2})^{\lambda_2}]^{1-\alpha}, \end{aligned}$$

due to Hölder's inequality applied for $p = 1/\alpha > 1$. Since ϵ was arbitrary, convexity of $\Lambda_Q\big|_{(-\infty,0)}$ follows from

$$\begin{aligned}\Lambda_Q(\alpha\lambda_1 + (1-\alpha)\lambda_2) &\leq \underline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[(X_T^{\tilde{\pi}})^{\alpha\lambda_1+(1-\alpha)\lambda_2}] \\ &\leq \alpha \overline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[(X_T^{\pi_1})^{\lambda_1}] + (1-\alpha) \overline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[(X_T^{\pi_2})^{\lambda_2}] \\ &\leq \alpha(\Lambda_Q(\lambda_1) + \epsilon) + (1-\alpha)(\Lambda_Q(\lambda_2) + \epsilon).\end{aligned}$$

Using that $\Lambda(\lambda) = \Lambda_{Q^{\eta^*(\lambda)}}(\lambda)$ this result applied to $Q := Q^{\eta^*(\alpha\lambda_1+(1-\alpha)\lambda_2)}$ yields

$$\begin{aligned}\Lambda(\alpha\lambda_1 + (1-\alpha)\lambda_2) &= \Lambda_Q(\alpha\lambda_1 + (1-\alpha)\lambda_2) \\ &\leq \alpha\Lambda_Q(\lambda_1) + (1-\alpha)\Lambda_Q(\lambda_2) \\ &\leq \alpha\Lambda(\lambda_1) + (1-\alpha)\Lambda(\lambda_2),\end{aligned}$$

i. e., $\lambda \mapsto \Lambda(\lambda)$ is convex on $(-\infty, 0)$. $\square$

Our next goal is to show the duality relation $\underline{J}_Q = -\underline{\Lambda}_Q^*$ for a single measure $Q = Q^\eta \in \mathcal{Q}$. This will follow by translating the main arguments used in Hata, Nagai and Sheu [18] for a linear Gaussian factor model to our general setting. To this end, we need the following regularity assumptions, summarized as

Assumption 4.1. *Let $\eta : \mathbb{R} \to \Gamma$ be a measurable function, and let $Q = Q^\eta \in \mathcal{Q}$ be the measure associated with the feedback control $\eta_t = \eta(Y_t)$, $t \geq 0$. Assume furthermore that the ergodic Bellman equation*

$$\tilde{\Lambda}_Q(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy} + \tfrac{1}{1-\lambda}\varphi_y^2] + \sup_{\nu\in\mathbb{R}}\{(1-\lambda)l(\lambda, \eta(\cdot), \nu, \cdot) + \varphi_y h(\lambda, \eta(\cdot), \nu, \cdot)\} \tag{65}$$

has for any $\lambda < 0$ a solution $\tilde{\Lambda}_Q(\lambda) \in \mathbb{R}_-$, $\varphi(\lambda, \cdot) \in C^2(\mathbb{R})$ satisfying the conditions:

a) *There exists a constant $C_1(\lambda)$ such that $|\varphi_y(\lambda, y)| \leq C_1(\lambda)(1 + |y|)$.*

b) *The solutions $\tilde{\Lambda}_Q(\lambda), \varphi(\lambda, \cdot)$ are continuously differentiable in λ on $(-\infty, 0)$, and it holds that*

$$|\tfrac{d}{d\lambda}\varphi_y(\lambda, y)| \leq C_2(\lambda)(1 + |y|) \quad \textit{for some constant } C_2(\lambda) > 0.$$

c) *The function $\kappa_1(\lambda, \cdot)$ defined by*

$$\begin{aligned}\kappa_1(\lambda, y) := g(y) &+ (\rho, \eta^{1\cdot}(y)y + \eta^{2\cdot}(y)) \\ &+ \tfrac{\lambda}{1-\lambda}\rho_1[\theta(y) + \eta^{11}(y)y + \eta^{21}(y)] + (\tfrac{1}{1-\lambda}\rho_1^2 + \rho_2^2)\varphi_y(\lambda, y)\end{aligned}$$

satisfies $y\kappa_1(\lambda, y) \leq -C_3(\lambda)y^2 + C_4(\lambda)$ with constants $C_3(\lambda), C_4(\lambda) > 0$.

d) *Let R be the probability measure such that $dY_t = \kappa_1(\lambda, Y_t)\,dt + \rho\,dB_t$, where B denotes a two-dimensional Brownian motion under R. Then*

$$\lim_{T\uparrow\infty} \tfrac{1}{T} \ln E_R[e^{-\varphi(\lambda, Y_T)}] = 0.$$

The following theorem will show that the convex function $\Lambda_Q|_{(-\infty,0)}$ in (64) coincides with the function $\tilde{\Lambda}_Q$ given by the solutions of (65). Thus Λ_Q has the regularity property required in part b) of Assumption 4.1. This implies that the Fenchel-Legendre transform $\underline{\Lambda}^*_Q(c) = \sup_{\lambda<0}\{\lambda c - \Lambda_Q(\lambda)\}$, $c \in \mathbb{R}$, is given by

$$\underline{\Lambda}^*_Q(c) = \begin{cases} \infty & \text{for } c < \Lambda'_Q(-\infty), \\ \lambda[c]\Lambda'_Q(\lambda[c]) - \Lambda_Q(\lambda[c]) & \text{for } \Lambda'_Q(-\infty) < c < \Lambda'_Q(0), . \\ 0 & \text{for } c \geq \Lambda'_Q(0). \end{cases} \tag{66}$$

Here we use the notation $\Lambda'_Q(-\infty) := \lim_{\lambda\downarrow-\infty}\Lambda'_Q(\lambda)$, $\Lambda'_Q(0) := \lim_{\lambda\uparrow 0}\Lambda'_Q(\lambda)$, and $\lambda[c] \in (-\infty, 0)$ is taken such that $\Lambda'_Q(\lambda[c]) = c \in (\Lambda'_Q(-\infty), \Lambda'_Q(0))$ (first order condition for the maximum). Moreover, $\underline{\Lambda}^*_Q$ is continuous on $(\Lambda'_Q(-\infty), \infty)$.

Theorem 4.1. *Consider the model $Q = Q^\eta \in \mathcal{Q}$, and assume that Assumption 4.1 is satisfied for Q. Then we have:*

i) *For any $\lambda < 0$, the value $\tilde{\Lambda}_Q(\lambda)$ given by the solution to the ergodic Bellman equation (65) coincides with the optimal growth rate for expected power utility*

$$\Lambda_Q(\lambda) = \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[(X^\pi_T)^\lambda]$$

Moreover, the control $\pi^{*,\eta}_t(\lambda) := \pi^{*,\eta}(\lambda, Y_t)$, $t \geq 0$, defined by the function

$$\pi^{*,\eta}(\lambda, y) := \tfrac{1}{1-\lambda}\tfrac{1}{\sigma}(\varphi_y(\lambda, y)\rho_1 + \theta(y) + \eta^{11}(y)y + \eta^{21}(y)), \tag{67}$$

achieves the infimum among all investment strategies $\pi \in \mathcal{A}$, i. e.,

$$\Lambda_Q(\lambda) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[(X^{\pi^{*,\eta}(\lambda)}_T)^\lambda] \tag{68}$$

ii) *For any $c \neq \Lambda'_Q(-\infty)$, the Fenchel-Legendre transform $\underline{\Lambda}^*_Q$ in (66) yields the optimal rate of decay for downside risk, i. e.,*

$$\underline{J}_Q(c) = \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln Q[L^\pi_T \leq c] = -\underline{\Lambda}^*_Q(c).$$

iii) *For any $c \in (\Lambda'_Q(-\infty)), \Lambda'_Q(0))$, the feedback control $\widehat{\pi}^{c,\eta} := \pi^{*,\eta}(\lambda[c])$ minimizes the asymptotic probability of falling below the threshold c, i. e.,*

$$\underline{J}_Q(c) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln Q[L^{\widehat{\pi}^{c,\eta}}_T \leq c].$$

iv) *For any $c < \Lambda'_Q(-\infty)$, the trading strategies $\widehat{\pi}^{n,\eta} := \pi^{*,\eta}(\lambda[\Lambda'_Q(-\infty) + 1/n])$, $n \in \mathbb{N}$, yield*

$$\lim_{n\uparrow\infty} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln Q[L^{\widehat{\pi}^{n,\eta}}_T \leq c] = -\infty = \underline{J}_Q(c). \tag{69}$$

Proof. 1) We shall derive i) as a special case of Theorem 3.2. To this end, we recall that the dynamics of Y, S^1 under $Q = Q^\eta$ is given by (8), replace Q_0 by the new reference measure $\widetilde{Q}_0 = Q$ and take as initial data

$$\widetilde{g}(y) := g(y) + (\rho, \eta^{1\cdot}(y)y + \eta^{2\cdot}(y)), \quad \widetilde{\rho}_i := \rho_i, \quad i = 1, 2, \quad \widetilde{\sigma} := \sigma,$$
$$\widetilde{m}(y) := m(y) + \sigma(\eta^{11}(y)y + \eta^{21}(y)), \quad \widetilde{\theta}(y) := \theta(y) + \eta^{11}(y)y + \eta^{21}(y),$$

as well as the one-point set $\widetilde{\Gamma} := \{(0,0,0,0)\}$. This corresponds to the non-robust case $\mathcal{Q} := \{\widetilde{Q}_0\}$. In view of a), c) and d) in Assumption 4.1 the solution $(\tilde{\Lambda}_Q(\lambda), \varphi(\lambda,\cdot))$ also satisfies Assumption 3.1. Theorem 3.2 applied to the one-point set $\mathcal{Q} = \{\widetilde{Q}_0\}$ yields that the value $\tilde{\Lambda}_Q(\lambda)$ coincides with the optimal growth rate

$$\inf_{\pi\in\mathcal{A}} \underline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{\widetilde{Q}\in\mathcal{Q}} E_{\widetilde{Q}}[(X_T^\pi)^\lambda] = \inf_{\pi\in\mathcal{A}} \underline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln E_Q[(X_T^\pi)^\lambda] = \Lambda_Q(\lambda)$$

for power utility with parameter $\lambda < 0$. Moreover, the optimality statement (68) for $\pi^{*,\eta}(\lambda)$ follows from (61) and (62), since here $\eta^*(\lambda,y) \in \widetilde{\Gamma} = \{(0,0,0,0)\}$.

2) Let us now focus on part ii). Since $\underline{J}_Q \leq -\underline{\Lambda}_Q^*$, and since $\underline{\Lambda}_Q^*(c) = \infty$ for any $c < \Lambda_Q'(-\infty)$, it suffices to show

$$\underline{J}(c) = \inf_{\pi\in\mathcal{A}} \underline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln Q[L_T^\pi \leq c] \geq -\underline{\Lambda}_Q^*(c) \quad \text{for any } c > \Lambda_Q'(-\infty). \tag{70}$$

To this end, we first fix $c \in (\Lambda_Q'(-\infty), \Lambda_Q'(0)]$ and take $\epsilon > 0$ small enough such that $c - \epsilon > \Lambda_Q'(-\infty)$. In that case,

$$\underline{\Lambda}_Q^*(c-\epsilon) = \lambda[c-\epsilon](c-\epsilon) - \Lambda_Q(\lambda[c-\epsilon]) = \lambda[c-\epsilon]\Lambda_Q'(\lambda[c-\epsilon]) - \Lambda_Q(\lambda[c-\epsilon]), \tag{71}$$

due to $\Lambda_Q'(\lambda[c-\epsilon]) = c-\epsilon$. In order to compute Λ_Q', we shall now differentiate both sides of (65). For this purpose, note that this EBE takes the explicit form

$$\begin{aligned}\Lambda_Q(\lambda) = \tfrac{1}{2}\|\rho\|^2[\varphi_{yy}(\lambda,y) + \tfrac{1}{1-\lambda}\varphi_y^2(\lambda,y)] - \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}\rho_2^2(y)\varphi_y^2(\lambda,y) \\ + \varphi_y(\lambda,y)(g(y) + (\rho, \eta^{1\cdot}(y)y + \eta^{2\cdot}(y)) + \tfrac{\lambda}{1-\lambda}\rho_1[\theta(y) + \eta^{11}(y)y + \eta^{21}(y)]) \\ + \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}[\theta(y) + \eta^{11}(y)y + \eta^{21}(y)]^2 + \lambda r(y).\end{aligned}$$

To keep the notation as simple as possible, we write for short λ instead of $\lambda[c-\epsilon]$, $\varphi(y) = \varphi(\lambda,y)$ and $\gamma(y) := \frac{d}{d\mu}\varphi(\mu,y)|_{\mu=\lambda}$. Then it follows that

$$\Lambda_Q'(\lambda) = \tfrac{1}{2}\|\rho\|^2\gamma_{yy}(y) + \kappa_1(\lambda,y)\gamma_y(y) + \kappa_2(\lambda,y). \tag{72}$$

Here we use the functions $\kappa_1(\lambda,\cdot)$, defined in Assumption 4.1 c), and

$$\kappa_2(\lambda,y) := \tfrac{1}{2}(\sigma\pi^{*,\eta}(\lambda,y))^2 + r(y). \tag{73}$$

On the other hand, we can rewrite (65) as

$$\Lambda_Q(\lambda) = \tfrac{1}{2}\|\rho\|^2\varphi_{yy}(y) + \kappa_1(\lambda,y)\varphi_y(y) + \kappa_3(\lambda,y)$$

with

$$\kappa_3(\lambda,y) := -\tfrac{1}{2}[\tfrac{1}{1-\lambda}\rho_1^2(y) + \rho_2^2(y)]\varphi_y^2(y) + \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}[\theta(y) + \eta^{11}(y)y + \eta^{21}(y)]^2 + \lambda r(y).$$

Thus the right-hand side of (71) takes the form

$$\begin{aligned}\Lambda_Q(\lambda) - \lambda\Lambda_Q'(\lambda) = \tfrac{1}{2}\|\rho\|^2(\varphi_{yy} - \lambda\gamma_{yy})(y) + \kappa_1(\lambda,y)(\varphi_y - \lambda\gamma_y)(y) \\ + \kappa_3(\lambda,y) - \lambda\kappa_2(\lambda,y).\end{aligned} \tag{74}$$

Note also that

$$\kappa_3(\lambda,y) - \lambda\kappa_2(\lambda,y) = -\tfrac{1}{2}[\tfrac{1}{1-\lambda}(\rho_1\varphi_y(y) + \lambda[\theta(y) + \eta^{11}(y)y + \eta^{21}(y)])]^2 - \tfrac{1}{2}[\rho_2\varphi_y(y)]^2.$$

Let us now turn to the analysis of the wealth X_T^π generated by a strategy $\pi \in \mathcal{A}$. In view of the SDE (9) we have

$$X_T^\pi = x_0 e^{\int_0^T \sigma\pi_t\, dW_t^{1,\eta} + \int_0^T (r(Y_t)+[\theta(Y_t)+\eta_t^{11}Y_t+\eta_t^{21}]\sigma\pi_t - \frac{1}{2}\sigma^2\pi_t^2)\, dt}. \tag{75}$$

Our aim is to connect X_T^π and the exponential growth rate $L_T^\pi = \frac{1}{T}\ln X_T^\pi$ with the expression κ_2 and with the feedback control $\widehat{\pi}^{c-\epsilon,\eta} = \pi^{*,\eta}(\lambda)$ defined by (67) for $\lambda = \lambda[c-\epsilon]$. Using (73), an easy computation yields for the term inside the second integral in (75)

$$\begin{aligned} r(y) + [\theta(y) + \eta^{11}(y)y + \eta^{21}(y)]\sigma\pi - \tfrac{1}{2}\sigma^2\pi^2 &= -\tfrac{1}{2}\sigma^2(\pi - \pi^{*,\eta}(\lambda,y))^2 \\ &\quad - \sigma\pi[\tfrac{1}{1-\lambda}(\rho_1\varphi_y(y) + \lambda[\theta(y) + \eta^{11}(y)y + \eta^{21}(y)])] + \kappa_2(\lambda, y), \end{aligned}$$

i. e.,

$$X_T^\pi = x_0 e^{\int_0^T \sigma\pi_t (dW_t^{1,\eta} - \frac{1}{1-\lambda}(\rho_1\varphi_y(Y_t)+\lambda[\theta(Y_t)+\eta_t^{11}Y_t+\eta_t^{21}])\, dt)} e^{\int_0^T \kappa_2(\lambda,Y_t) - \frac{1}{2}\sigma^2(\pi_t - \pi_t^{*,\eta}(\lambda))^2\, dt}.$$

To eliminate the drift appearing in the first integral, we introduce the new probability measure R on $(\Omega, \mathcal{F}_T)$ via

$$\tfrac{dR}{dQ}\big|_{\mathcal{F}_T} := \mathcal{E}\Big(\int_0^\cdot \tfrac{\rho_1\varphi_y(Y_t)+\lambda[\theta(Y_t)+\eta_t^{11}Y_t+\eta_t^{21}]}{1-\lambda}\, dW_t^{1,\eta} + \int_0^\cdot \rho_2\varphi_y(Y_t)\, dW_t^{2,\eta}\Big)_T.$$

Girsanov's theorem ensures that $B = (B_t^1, B_t^2)_{t\le T}$ defined by

$$\begin{aligned} B_t^1 &:= W_t^{1,\eta} - \int_0^t \tfrac{1}{1-\lambda}(\varphi_y(Y_s)\rho_1(Y_s) + \lambda[\theta(Y_s) + \eta_s^{11}Y_s + \eta_s^{21}])\, ds, \\ B_t^2 &:= W_t^{2,\eta} - \int_0^t \rho_2(Y_s)\varphi_y(Y_s)\, ds \end{aligned} \tag{76}$$

is a two-dimensional R-Brownian motion. Thus the exponential growth rate $L_T^\pi = \frac{1}{T}\ln X_T^\pi$ admits the representation

$$\begin{aligned} L_T^\pi &= \tfrac{1}{T}\Big(\ln x_0 + \int_0^T \sigma\pi_t\, dB_t^1 + \int_0^T \kappa_2(\lambda, Y_t) - \tfrac{1}{2}\sigma^2(\pi_t - \pi_t^{*,\eta}(\lambda))^2\, dt\Big) \\ &= \tfrac{1}{T}\Big(\ln x_0 + \int_0^T \sigma\pi_t^{*,\eta}(\lambda)\, dB_t^1 + \ln\mathcal{E}\Big(\int_0^\cdot (\sigma(\pi_t - \pi_t^{*,\eta}(\lambda))\, dB_t^1\Big)_T + \int_0^T \kappa_2(\lambda, Y_t)\, dt\Big). \end{aligned}$$

In a next step, we consider the shortfall probability

$$Q[L_T^\pi \le c] = E_R[(\tfrac{dR}{dQ}\big|_{\mathcal{F}_T})^{-1}; L_T^\pi \le c],$$

where the density term $(dR/dQ|_{\mathcal{F}_T})^{-1}$ can be rewritten as

$$(\tfrac{dR}{dQ}\big|_{\mathcal{F}_T})^{-1} = e^{-\int_0^T \frac{1}{1-\lambda}(\rho_1\varphi_y(Y_t)+\lambda[\theta(Y_t)+\eta_t^{11}Y_t+\eta_t^{21}])\, dB_t^1 - \int_0^T \rho_2\varphi_y(Y_t)\, dB_t^2 + \int_0^T \kappa_3(\lambda,Y_t) - \lambda\kappa_2(\lambda,Y_t)\, dt}.$$

In order to obtain an appropriate lower bound, we introduce the following events:

$$
\begin{aligned}
E_{1,T} :=& \{\tfrac{1}{T}\int_0^T \kappa_3(\lambda, Y_t) - \lambda\kappa_2(\lambda, Y_t)\, dt \geq \Lambda_Q(\lambda) - \lambda\Lambda_Q'(\lambda) - \epsilon\},\\
E_{2,T} :=& \{\tfrac{1}{T}\int_0^T \tfrac{\rho_1\varphi_y(Y_t)+\lambda[\theta(Y_t)+\eta_t^{11}Y_t+\eta_t^{21}]}{1-\lambda}\, dB_t^1 + \tfrac{1}{T}\int_0^T \rho_2\varphi_y(Y_t)\, dB_t^2 \leq \epsilon\},\\
E_{3,T} :=& \{\tfrac{1}{T}(\ln x_0 + \int_0^T \kappa_2(\lambda, Y_t)\, dt) \leq \Lambda_Q'(\lambda) + \tfrac{\epsilon}{3}\},\\
E_{4,T} :=& \{\tfrac{1}{T}\int_0^T \sigma\pi_t^{*,\eta}(\lambda)\, dB_t^1 \leq \tfrac{\epsilon}{3}\},\\
E_{5,T} :=& \{\tfrac{1}{T}\ln \mathcal{E}(\int_0^\cdot \sigma(\pi_t - \pi_t^{*,\eta}(\lambda))\, dB_t^1)_T \leq \tfrac{\epsilon}{3}\}.
\end{aligned}
$$

Then

$$E_{1,T} \cap E_{2,T} \subseteq \{(\tfrac{dR}{dQ}\big|_{\mathcal{F}_T})^{-1} \geq \exp((\Lambda_Q(\lambda) - \lambda\Lambda_Q'(\lambda) - 2\epsilon)T)\},$$

and

$$E_{3,T} \cap E_{4,T} \cap E_{5,T} \subseteq \{L_T^\pi \leq c\},$$

due to $\Lambda_Q'(\lambda) = \Lambda_Q'(\lambda[c-\epsilon]) = c - \epsilon$. In step 4) we will show that

$$\exists\, T(\epsilon)\ \forall\, T \geq T(\epsilon): \quad R[E_{i,T}^c] \leq \epsilon \quad \text{for all } \pi \in \mathcal{A},\ i \in \{1, \ldots, 5\}. \tag{77}$$

This allows us to conclude

$$
\begin{aligned}
Q[L_T^\pi \leq c] &= E_R[(\tfrac{dR}{dQ}\big|_{\mathcal{F}_T})^{-1}; L_T^\pi \leq c]\\
&\geq E_R[(\tfrac{dR}{dQ}\big|_{\mathcal{F}_T})^{-1}; E_{1,T} \cap E_{2,T} \cap \{L_T^\pi \leq c\}]\\
&\geq \exp((-\lambda\Lambda_Q'(\lambda) + \Lambda_Q(\lambda) - 2\epsilon)T) R[E_{1,T} \cap E_{2,T} \cap \{L_T^\pi \leq c\}]\\
&\geq \exp((-\lambda\Lambda_Q'(\lambda) + \Lambda_Q(\lambda) - 2\epsilon)T) R[\cap_{i=1}^5 E_{i,T}]\\
&\geq \exp((-\lambda\Lambda_Q'(\lambda) + \Lambda_Q(\lambda) - 2\epsilon)T)(1 - 5\epsilon)
\end{aligned}
$$

as soon as $T \geq T(\epsilon)$. Note that this lower bound is uniform among all $\pi \in \mathcal{A}$. Passing to the limit as T goes to infinity, we obtain

$$\underline{J}_Q(c) \geq \varliminf_{T\uparrow\infty} \tfrac{1}{T}\ln(\exp((-\lambda\Lambda_Q'(\lambda) + \Lambda_Q(\lambda) - 2\epsilon)T)(1-5\epsilon)) = -\lambda\Lambda_Q'(\lambda) + \Lambda_Q(\lambda) - 2\epsilon.$$

But in view of (71) the last inequalty is equivalent to

$$\underline{J}_Q(c) \geq -\underline{\Lambda}_Q^*(c - \epsilon) - 2\epsilon.$$

Sending ϵ to zero and using the continuity of $\underline{\Lambda}_Q^*$ at c, we finally get the desired estimate $\underline{J}_Q(c) \geq -\underline{\Lambda}_Q^*(c)$ for any $c \in (\Lambda_Q'(-\infty), \Lambda_Q'(0)]$.

It remains to consider the case $c > \Lambda_Q'(0)$. Using monotonicity of $\underline{J}_Q$ and the preceding result for $\underline{\Lambda}_Q'(0)$, we obtain

$$0 \geq \underline{J}_Q(c) \geq \underline{J}_Q(\Lambda_Q'(0)) = -\underline{\Lambda}_Q^*(\Lambda_Q'(0)) = 0 = -\underline{\Lambda}_Q^*(c).$$

We have thus shown (70). This completes the proof of $\underline{J}_Q(c) = -\underline{\Lambda}_Q^*(c)$ for all $c \neq \Lambda_Q'(-\infty)$.

3) Let us now identify trading strategies which minimize the asymptotic probability of falling below the target rate c.

For $c \in (\Lambda_Q'(-\infty), \Lambda_Q'(0))$, optimality of the strategy $\widehat{\pi}^{c,\eta} = \pi^{*,\eta}(\lambda[c])$ follows by combining the duality relation $\underline{J}_Q(c) = -\underline{\Lambda}_Q^*(c)$ (cf. part 2)) with the estimate

$$\begin{aligned}
\underline{J}_Q(c) &\leq \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln Q[L_T^{\widehat{\pi}^{c,\eta}} \leq c] \\
&= \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln Q[(X_T^{\widehat{\pi}^{c,\eta}})^{\lambda[c]} \geq e^{\lambda[c]cT}] \\
&\leq -\lambda[c]c + \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln E_Q[(X_T^{\pi^{*,\eta}(\lambda[c])})^{\lambda[c]}] \\
&= -\lambda[c]c + \Lambda_Q(\lambda[c]) = -\underline{\Lambda}_Q^*(c).
\end{aligned}$$

Here we have used (68) in order to obtain the first equality in the last line.

If $c < \Lambda_Q'(-\infty)$, then repeating the previous argument for the trading strategies $\widehat{\pi}^{n,\eta} = \pi^{*,\eta}(\lambda_n)$, $\lambda_n := \lambda[\Lambda_Q'(-\infty) + 1/n]$, $n \in \mathbb{N}$, leads to

$$\underline{J}(c) \leq \lim_{n\uparrow\infty} \underline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln Q[L_T^{\widehat{\pi}^{n,\eta}} \leq c] \leq \lim_{n\uparrow\infty} (-\lambda_n c + \Lambda_Q(\lambda_n)) = -\infty.$$

To verify the last equality, we take some reference point $\lambda < 0$. Then convexity of $\Lambda_Q|_{(-\infty,0)}$ ensures that

$$\Lambda_Q'(\lambda_n)(\lambda - \lambda_n) \leq \Lambda_Q(\lambda) - \Lambda_Q(\lambda_n) \quad \text{for all } \lambda_n < \lambda.$$

Since λ_n tends to $-\infty$ as $n \uparrow \infty$ and $\lim_{n\uparrow\infty} \Lambda_Q'(\lambda_n) = \Lambda_Q'(-\infty) > c$, this implies

$$\lim_{n\uparrow\infty} (-\lambda_n c + \Lambda_Q(\lambda_n)) \leq \lim_{n\uparrow\infty} (\Lambda_Q(\lambda) - \Lambda_Q'(\lambda_n)\lambda + \lambda_n(\Lambda_Q'(\lambda_n) - c)) = -\infty.$$

On the other hand, we have $\underline{J}_Q(c) = -\underline{\Lambda}_Q^*(c) = -\infty$, due to part 2) and (66). This shows (69).

4) It remains to verify the asymptotic estimates in (77). To this end, note that the dynamics of the factor process Y under R is given by the SDE

$$dY_t = [g(Y_t) + (\rho, \eta^{1\cdot}(Y_t)Y_t + \eta^{2\cdot}(Y_t))]\, dt + \rho\, dW_t^\eta = \kappa_1(\lambda, Y_t)\, dt + \rho\, dB_t$$

with initial value $Y_0 = y_0$. Here B denotes the two-dimensional R-Brownian motion defined in (76), and $\kappa_1(\lambda, \cdot)$ is the corresponding drift function introduced in Assumption 4.1 c).

First, by combining Tchebychev's inequality, (74) and Itô's formula applied to $\varphi - \lambda\gamma$, we get

$$\begin{aligned}
&R[E_{1,T}^c] \leq R[|\Lambda_Q(\lambda) - \lambda\Lambda_Q'(\lambda) - \tfrac{1}{T} \int_0^T \kappa_3(\lambda, Y_t) - \lambda\kappa_2(\lambda, Y_t)\, dt| \geq \epsilon] \\
&\leq \epsilon^{-2} E_R[|\Lambda_Q(\lambda) - \lambda\Lambda_Q'(\lambda) - \tfrac{1}{T} \int_0^T \kappa_3(\lambda, Y_t) - \lambda\kappa_2(\lambda, Y_t)\, dt|^2] \\
&= (\epsilon T)^{-2} E_R[|\int_0^T (\varphi_y - \lambda\gamma_y)(Y_t)\kappa_1(\lambda, Y_t)\, dt + \tfrac{1}{2} \int_0^T (\varphi_{yy} - \lambda\gamma_{yy})(Y_t)\|\rho\|^2\, dt|^2] \\
&= (\epsilon T)^{-2} E_R[|(\varphi - \lambda\gamma)(Y_T) - (\varphi - \lambda\gamma)(y_0) - \int_0^T (\varphi_y - \lambda\gamma_y)(Y_t)\rho\, dB_t|^2]. \qquad (78)
\end{aligned}$$

Next, by a) and b) in Assumption 4.1, we may find a constant $C_5(\lambda) > 0$ such that

$$|(\varphi - \lambda\gamma)(y)| \leq C_5(\lambda)(1 + y^2) \quad \text{and} \quad |(\varphi_y - \lambda\gamma_y)(y)| \leq C_5(\lambda)(1 + |y|) \qquad (79)$$

Moreover, by Assumption 4.1 c), Lemma 8.2 in [24] ensures that

$$\exists\, C_6(\lambda) > 0\; \forall\, T \geq 0: \; E_R[Y_T^2] \leq C_6(\lambda)(1+y_0^2), E_R[Y_T^4] \leq C_6(\lambda)(1+y_0^4). \qquad (80)$$

Thus the stochastic integral in (78) is a square integrable R-martingale, due to

$$\begin{aligned} E_R[\langle \int_0^{\cdot} (\varphi_y - \lambda\gamma_y)(Y_t)\rho\, dB_t\rangle_T] &= E_R[\int_0^T ((\varphi_y - \lambda\gamma_y)(Y_t))^2 \|\rho\|^2\, dt] \\ &\leq 2C_5^2(\lambda)\|\rho\|^2(1+\sup_{t\leq T} E_R[Y_t^2])T < \infty. \end{aligned}$$

Going on with estimate (78), it follows by Itô's isometry, (79) and (80) that

$$\begin{aligned} R[E_{1,T}^c] &\leq (\epsilon T)^{-2}(|(\varphi-\lambda\gamma)(y_0)|^2 + E_R[|(\varphi-\lambda\gamma)(Y_T)|^2]) \\ &\quad + (\epsilon T)^{-2} E_R[\int_0^T |(\varphi_y-\lambda\gamma_y)(Y_t)|^2\|\rho\|^2\, dt] \\ &= (\epsilon T)^{-2} 2(C_5(\lambda))^2[1+y_0^4+1+E_R[Y_T^4]+T\|\rho\|^2(1+\sup_{t\leq T} E_R[Y_t^2])] \\ &\leq (\epsilon T)^{-2} 2(C_5(\lambda))^2[2+y_0^4+C_6(\lambda)(1+y_0^4)+T\|\rho\|^2(1+C_6(\lambda)(1+y_0^2))]. \end{aligned}$$

But this translates into $R[E_{1,T}^c] \leq \epsilon$ as soon as T exceeds some $T(\epsilon)$.

Using (72) and $|\gamma_y(y)| \leq C_2(\lambda)(1+|y|)$, the same arguments lead to

$$\begin{aligned} R[E_{3,T}^c] &\leq R[|\tfrac{1}{T}(\ln x_0 + \int_0^T \kappa_2(\lambda, Y_t)\, dt) - \Lambda_Q'(\lambda)| \geq \tfrac{\epsilon}{3}] \\ &\leq (\tfrac{\epsilon}{3})^{-2} E_R[|\tfrac{1}{T}(\ln x_0 + \int_0^T \kappa_2(\lambda, Y_t)\, dt) - \Lambda_Q'(\lambda)|^2] \\ &= (\tfrac{\epsilon}{3}T)^{-2} E_R[|\ln x_0 - \int_0^T \gamma_y(Y_t)\kappa_1(\lambda, Y_t)\, dt - \tfrac{1}{2}\int_0^T \gamma_{yy}(Y_t)\|\rho\|^2\, dt|^2] \\ &= (\tfrac{\epsilon}{3}T)^{-2} E_R[|\ln x_0 - \gamma(Y_T) + \gamma(y_0) + \int_0^T \gamma_y(Y_t)\rho\, dB_t|^2] \leq \text{const.}/T \end{aligned}$$

i. e., $R[E_{3,T}^c] \leq \epsilon$ as soon as T is large enough.

To estimate $R[E_{2,T}^c]$, notice that by Assumption 2.1, Assumption 4.1 a), and boundedness of $\eta^{11}(\cdot)$, $\eta^{21}(\cdot)$ there exists an constant $C_8(\lambda)$ with

$$\begin{aligned} |\tfrac{1}{1-\lambda}(\rho_1\varphi_y(y) + \lambda[\theta(y) + \eta^{11}(y)y + \eta^{21}(y)])| &\leq C_8(\lambda)(1+|y|), \\ |\rho_2\varphi_y(y)| &\leq C_8(\lambda)(1+|y|). \end{aligned}$$

Tchebychev's inequality combined with Itô's isometry thus implies

$$\begin{aligned} R[E_{2,T}^c] &\leq (\epsilon T)^{-2} E_R[|\int_0^T \tfrac{\rho_1\varphi_y(Y_t)+\lambda[\theta(Y_t)+\eta_t^{11}Y_t+\eta_t^{21}]}{1-\lambda}\, dB_t^1 + \int_0^T \rho_2\varphi_y(Y_t)\, dB_t^2|^2] \\ &= (\epsilon T)^{-2} E_R[\int_0^T |\tfrac{\rho_1\varphi_y(Y_t)+\lambda[\theta(Y_t)+\eta_t^{11}Y_t+\eta_t^{21}]}{1-\lambda}|^2 + |\rho_2\varphi_y(Y_t)|^2\, dt] \\ &\leq (\epsilon T)^{-2} 4(C_8(\lambda))^2 T(1+\sup_{t\leq T} E_R[Y_t^2]) \\ &\leq \tfrac{1}{\epsilon^2 T} 4(C_8(\lambda))^2(1+C_6(\lambda)(1+y_0^2)), \end{aligned}$$

and this leads to $R[E_{2,T}^c] \leq \epsilon$ for $T \geq T(\epsilon)$. Since $|\sigma\pi^{*,\eta}(\lambda, y)| \leq C_9(\lambda)(1+|y|)$, a similar argument yields $R[E_{4,T}^c] \leq \epsilon$ for T large enough.

Finally, using once more Tchebychev's inequality, the supermartingale property of the Itô exponential $\mathcal{E}(\cdot)$ yields

$$R[E^c_{5,T}] \le \exp(-\tfrac{\epsilon}{3}T) E_R[\mathcal{E}(\int_0^\cdot \sigma(\pi_t - \pi_t^{*,\eta}(\lambda))\, dB^1_t)_T] \le \exp(-\tfrac{\epsilon}{3}T).$$

We have thus shown (77), and this closes the gap in part 2). □

With these results for the non-robust case we are now ready to analyze the asymptotic minimization of robust downside risk. Notice that the following proof does no longer involve the specific structure of our model; it only uses the identification $\Lambda(\lambda) = \Lambda_{Q^{\eta^*(\lambda)}}(\lambda)$.

Recall from Proposition 4.1 that the function $\Lambda|_{(-\infty,0)}$ defined by (15) is convex. If Λ is even continuously differentiable on $(-\infty, 0)$, then the Fenchel-Legendre transform $\underline{\Lambda}^*$ in (16) can be computed as

$$\underline{\Lambda}^*(c) = \begin{cases} \infty & \text{for } c < \Lambda'(-\infty) \\ \lambda[c]\Lambda'(\lambda[c]) - \Lambda(\lambda[c]) & \text{for } \Lambda'(-\infty) < c < \Lambda'(0) \\ 0 & \text{for } c \ge \Lambda'(0) \end{cases} . \tag{81}$$

Here we use the notation $\Lambda'(-\infty) := \lim_{\lambda\downarrow-\infty} \Lambda'(\lambda)$, $\Lambda'(0) := \lim_{\lambda\uparrow 0} \Lambda'(\lambda)$, and $\lambda[c] < 0$ is chosen such that $\Lambda'(\lambda[c]) = c \in (\Lambda'(-\infty), \Lambda'(0))$. Moreover, $\underline{\Lambda}^*$ is continuous on $(\Lambda'(-\infty), \infty)$.

Theorem 4.2. *Suppose that the function $\Lambda|_{(-\infty,0)}$ defined by the optimal growth rates in (15) is continuously differentiable. For any $\lambda < 0$, we assume that Assumption 4.1 is satisfied for the worst-case measure $Q = Q^{\eta^*(\lambda)}$. Then we get:*

i) *The duality relation*

$$\underline{J}(c) := \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} Q^\eta[L^\pi_T \le c] = -\underline{\Lambda}^*(c) \tag{82}$$

holds for any $c \ne \Lambda'(-\infty)$.

ii) *For any $c \in (\Lambda'(-\infty), \Lambda'(0))$, the controls $\widehat{\pi}^c := \pi^*(\lambda[c])$ and $\widehat{\eta}^c := \eta^*(\lambda[c])$ yield the optimal strategy and the worst-case model $Q^{\widehat{\eta}^c}$ in the sense that*

$$\underline{J}(c) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} Q^\eta[L^{\widehat{\pi}^c}_T \le c] = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln Q^{\widehat{\eta}^c}[L^{\widehat{\pi}^c}_T \le c] = \underline{J}_{Q^{\widehat{\eta}^c}}(c).$$

Thus the "robust large deviations control problem" (12) reduces to a non-robust control problem with respect to the specific model $Q^{\widehat{\eta}^c} \in \mathcal{Q}$.

iii) *For any $c \ge \Lambda'(0)$, the sequence of probabilistic models $Q^{\widehat{\eta}^n} \in \mathcal{Q}$, $n \in \mathbb{N}$, associated with the feedback controls $\widehat{\eta}^n := \eta^*(\lambda[\Lambda'(0) - 1/n])$ is nearly least favorable in the sense that*

$$0 = \lim_{n\uparrow\infty} \underline{J}_{Q^{\widehat{\eta}^n}}(c) = \underline{J}(c).$$

iv) *For any $c < \Lambda'(-\infty)$, the investment strategies $\widehat{\pi}^n := \pi^*(\lambda[\Lambda'(-\infty) + 1/n])$, $n \in \mathbb{N}$, satisfy*

$$\lim_{n\uparrow\infty} \varliminf_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} Q^\eta[L^{\widehat{\pi}^n}_T \le c] = -\infty = \underline{J}(c). \tag{83}$$

Proof. 1) In order to show the duality relation $\underline{J} = -\underline{\Lambda}^*$, let us first argue for $c \in (\Lambda'(-\infty), \Lambda'(0))$. In that case, the concave function $f(\lambda) := \lambda c - \Lambda(\lambda)$, $\lambda < 0$, achieves its maximum among all $\lambda < 0$ at $\lambda[c] \in (-\infty, 0)$. On the other hand, f is dominated by the concave function $g(\lambda) := \lambda c - \Lambda_{Q^{\widehat{\eta}^c}}(\lambda)$, and it holds that

$$f(\lambda[c]) = \lambda[c]c - \Lambda(\lambda[c]) = \lambda[c]c - \Lambda_{Q^{\eta^*(\lambda[c])}}(\lambda[c]) = \lambda[c]c - \Lambda_{Q^{\widehat{\eta}^c}}(\lambda[c]) = g(\lambda[c]).$$

Since f and g are assumed to be continuously differentiable, this tangency implies the first order condition $g'(\lambda[c]) = 0$, i. e., $c = \Lambda'_{Q^{\widehat{\eta}^c}}(\lambda[c]) > \Lambda'_{Q^{\widehat{\eta}^c}}(-\infty)$. In other words, $\lambda[c]$ is also the maximizer of g among all $\lambda < 0$. Thus we have

$$\underline{\Lambda}^*(c) = f(\lambda[c]) = g(\lambda[c]) = \sup_{\lambda<0}\{\lambda c - \Lambda_{Q^{\widehat{\eta}^c}}(\lambda)\} = \underline{\Lambda}^*_{Q^{\widehat{\eta}^c}}(c).$$

Applying Theorem 4.1 with respect to $Q^{\widehat{\eta}^c} = Q^{\eta^*(\lambda[c])}$ gives the estimate

$$\underline{J}(c) \geq \underline{J}_{Q^{\widehat{\eta}^c}}(c) = -\underline{\Lambda}^*_{Q^{\widehat{\eta}^c}}(c) = -\underline{\Lambda}^*(c). \tag{84}$$

By Proposition 2.1 this yields $\underline{J}(c) = -\underline{\Lambda}^*(c)$ for $c \in (\Lambda'(-\infty), \Lambda'(0))$. To identify $\widehat{\pi}^c$ as the optimal strategy and $Q^{\widehat{\eta}^c}$ as the worst-case measure, we note that

$$\begin{aligned}
\overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln Q^{\widehat{\eta}^c}[L_T^{\widehat{\pi}^c} \leq c] &\leq \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^{\widehat{\pi}^c} \leq c] \\
&= \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[e^{\lambda[c]TL_T^{\widehat{\pi}^c}} \geq e^{\lambda[c]cT}] \\
&\leq \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} \{e^{-\lambda[c]cT} E_{Q^\eta}[e^{\lambda[c]TL_T^{\widehat{\pi}^c}}]\} \\
&= -\lambda[c]c + \Lambda(\lambda[c]) = -\underline{\Lambda}^*(c).
\end{aligned} \tag{85}$$

Here the first equality in the last line follows from the fact that the strategy $\widehat{\pi}^c = \pi^*(\lambda[c])$ is optimal for robust power utility with parameter $\lambda[c] < 0$ in the sense of (62), i. e.,

$$\Lambda(\lambda[c]) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[(X_T^{\widehat{\pi}^c})^{\lambda[c]}] = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} E_{Q^\eta}[e^{\lambda[c]TL_T^{\widehat{\pi}^c}}].$$

Putting (84) and (85) together, we see that

$$\underline{J}(c) = \underline{J}_{Q^{\widehat{\eta}^c}}(c) = -\underline{\Lambda}^*(c) \quad \text{for any } c \in (\Lambda'(-\infty), \Lambda'(0)),$$

that the strategy $\widehat{\pi}^c$ minimizes the robust downside risk, and that it also minimizes the asymptotic downside risk with respect to $Q^{\widehat{\eta}^c}$. Thus we have shown ii).

2) For $c \geq \Lambda'(0)$, it holds that $f|_{\mathbb{R}_-} \leq \underline{\Lambda}^*(c) = 0$ and $\lim_{\lambda\uparrow 0} f(\lambda) = 0$. On the other hand, the concave function $g_n(\lambda) := \lambda c - \Lambda_{Q^{\widehat{\eta}^n}}(\lambda)$, $\lambda < 0$, dominates f, and it is tangent to f at $\lambda_n := \lambda[\Lambda'(0) - \frac{1}{n}]$, due to $\Lambda_{Q^{\widehat{\eta}^n}}(\lambda_n) = \Lambda(\lambda_n)$. Clearly, by concavity, we have $\max_{\lambda\leq\lambda_n} g_n(\lambda) = f(\lambda_n)$. Since $\lim_{n\uparrow\infty} \lambda_n = 0$ this implies

$$\begin{aligned}
\lim_{n\uparrow\infty} \underline{\Lambda}^*_{Q^{\widehat{\eta}^n}}(c) &= \lim_{n\uparrow\infty} \sup_{\lambda<0}\{\lambda c - \Lambda_{Q^{\widehat{\eta}^n}}(\lambda)\} = \lim_{n\uparrow\infty} \max_{\lambda\leq\lambda_n} g_n(\lambda) \\
&= \lim_{n\uparrow\infty} f(\lambda_n) = 0 = \underline{\Lambda}^*(c).
\end{aligned}$$

Using again Theorem 4.1 we now see that

$$\begin{aligned}
0 &\geq \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^\pi \leq c] \geq \underline{J}(c) \geq \lim_{n\uparrow\infty} \underline{J}_{Q^{\widehat{\eta}^n}}(c) \\
&= -\lim_{n\uparrow\infty} \underline{\Lambda}^*_{Q^{\widehat{\eta}^n}}(c) = -\underline{\Lambda}^*(c) = 0,
\end{aligned}$$

and this implies i) for $c \geq \Lambda'(0)$ and iii).
3) For $c < \Lambda'(-\infty)$, we have $\underline{J}(c) = -\underline{\Lambda}^*(c) = -\infty$, due to (81) and the a priori estimate $\underline{J} \leq -\underline{\Lambda}^*$. This concludes the proof of i). It remains to show iv). Writing $\tilde{\lambda}_n := \lambda[\Lambda'(-\infty) + 1/n]$ we see that

$$\begin{aligned}\lim_{n\uparrow\infty} \underline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^{\widehat{\pi}^n} \leq c] &= \lim_{n\uparrow\infty} \underline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[e^{\tilde{\lambda}_n T L_T^{\widehat{\pi}^n}} \geq e^{\tilde{\lambda}_n c T}] \\ &\leq \lim_{n\uparrow\infty} \underline{\lim}_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} \{e^{-\tilde{\lambda}_n c T} E_{Q^\eta}[e^{\tilde{\lambda}_n T L_T^{\widehat{\pi}^n}}]\} \\ &= \lim_{n\uparrow\infty} (-\tilde{\lambda}_n c + \Lambda(\tilde{\lambda}_n)).\end{aligned}$$

Observe next that for some reference point $\tilde{\lambda} < 0$

$$\Lambda'(\lambda)(\tilde{\lambda} - \lambda) \leq \Lambda(\tilde{\lambda}) - \Lambda(\lambda) \quad \text{for all } \lambda < \tilde{\lambda},$$

due to the convexity of $\Lambda|_{(-\infty,0)}$. But this leads to

$$\lim_{n\uparrow\infty} (-\tilde{\lambda}_n c + \Lambda(\tilde{\lambda}_n)) \leq \lim_{n\uparrow\infty} (\Lambda(\tilde{\lambda}) - \Lambda'(\tilde{\lambda}_n)\tilde{\lambda} + \tilde{\lambda}_n(\Lambda'(\tilde{\lambda}_n) - c)) = -\infty$$

since $\tilde{\lambda}_n$ converges to $-\infty$ as $n \uparrow \infty$ and $\lim_{n\uparrow\infty} \Lambda'(\tilde{\lambda}_n) = \Lambda'(-\infty) > c$. Thus we have shown (83). □

Remark 4.1. *For $c \in (\Lambda'(-\infty), \Lambda'(0))$, Theorem 4.2 yields that*

$$\inf_{\pi \in \mathcal{A}} \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^\pi \leq c] \approx \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^{\widehat{\pi}^c} \leq c] \approx \exp(-\underline{\Lambda}^*(c)T), \quad \textit{as } T \uparrow \infty,$$

with a rate $\underline{\Lambda}^*(c) \in (0, \infty)$. In other words, the minimal worst-case probability of falling below the level c decays exponentially to zero, and the long term strategy $\widehat{\pi}^c$ provides a good approximation of an optimal strategy for the initial finite horizon problem (10). For $c \geq \Lambda'(0)$, however, the asymptotic approach (12) leads to

$$\inf_{\pi \in \mathcal{A}} \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^\pi \leq c] \approx 1, \tag{86}$$

due to $\underline{J}(c) = 0$. Here our Ansatz does not describe the asymptotics of (86) accurately. Instead one should look at the exponential decay of the distance

$$1 - \inf_{\pi \in \mathcal{A}} \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^\pi \leq c] = \sup_{\pi \in \mathcal{A}} \inf_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^\pi \geq c]$$

and compute a long term strategy that

$$\textit{maximizes} \quad \overline{\lim_{T\uparrow\infty}} \tfrac{1}{T} \ln \inf_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^\pi \geq c] \quad \textit{among all } \pi \in \mathcal{A}.$$

This robust outperformance criterion for long term investors is analyzed in [23].

5 Case studies

5.1 Black-Scholes model with uncertain drift

Taking constant drift coefficients $r(y) \equiv r$, $m(y) \equiv m$, we obtain under the reference measure Q_0 the one-dimensional Black-Scholes model, i. e.,

$$dS_t^0 = S_t^0 \, r \, dt, \quad dS_t^1 = S_t^1(m \, dt + \sigma \, dW_t^1).$$

Now suppose that the investor is uncertain about the "true" future drift of S^1: any adapted drift process with values in some interval $[a, b]$, $-\infty < a \le m \le b < \infty$, is possible. This uncertainty in the choice of the drift can be expressed by the ansatz $\Gamma = \{(0,0)\} \times [\frac{a-m}{\sigma}, \frac{b-m}{\sigma}] \times \{0\}$ and the associated prior probabilistic models $\mathcal{Q}$ defined by (6). More precisely, each element $Q^\eta \in \mathcal{Q}$ corresponds to a drift perturbation of the following type (cf. equation (8b)):

$$dS_t^1 = S_t^1([m + \sigma\eta_t^{21}]\,dt + \sigma\,dW_t^{1,\eta}).$$

In this example the factor process Y plays no role. In particular, the function $\varphi(\lambda, \cdot)$ appearing in the heuristic separation of time and space variables (32) is constant, and its derivatives $\varphi_y(\lambda, \cdot)$, $\varphi_{yy}(\lambda, \cdot)$ vanish. The EBE (42) for the asymptotics of robust utility maximization thus takes the simplified form

$$\tilde{\Lambda}(\lambda) = \sup_{\nu\in\mathbb{R}} \sup_{\eta\in\Gamma} \{\tfrac{1}{2}\tfrac{\lambda}{1-\lambda}[(\theta + \eta^{21})^2 + \nu^2] + \lambda r\} = \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}(\theta + \eta^{21,*})^2 + \lambda r,$$

where $\theta := \frac{m-r}{\sigma}$ denotes the constant market price of risk, and where $\eta^{21,*} \in [\frac{a-m}{\sigma}, \frac{b-m}{\sigma}]$ is the unique minimizer of the absolute value $|\theta + \eta^{21}|$. Defining the constant controls $\nu_t^*(\lambda) := 0$ and $\eta_t^* := \eta_t^*(\lambda) := (0, 0, \eta^{21,*}, 0)$, $t \ge 0$, the verification Theorem 3.2 is valid without any additional conditions as in Assumptions 3.1 and 3.2. We thus obtain the following

Results for the asymptotics of robust expected power utility:

- The optimal growth rate for robust expected power utility is given by

  $$\Lambda(\lambda) = \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}(\theta + \eta^{21,*})^2 + \lambda r, \quad \lambda < 0,$$

  and it coincides with the optimal growth rate $\Lambda_{Q^{\eta^*}}(\lambda)$ for the specific model Q^{η^*}.

- Investing the constant proportions $\pi_t^*(\lambda) = \frac{1}{1-\lambda}\frac{1}{\sigma}(\theta + \eta^{21,*})$, $t \ge 0$, is optimal.

- The worst-case measure Q^{η^*} does not depend on the parameter λ.

Set $\gamma := \frac{1}{2}(\theta + \eta^{21,*})^2$. Then we get $\Lambda'(-\infty) = r$, $\Lambda'(0) = \gamma + r$, and $\lambda[c] = 1 - \sqrt{\gamma/(c - r)}$ for any $c \in (r, \gamma + r)$, and so the Fenchel-Legendre transform $\underline{\Lambda}^*$ in (81) takes the explicit form

$$\underline{\Lambda}^*(c) = \begin{cases} \infty & \text{for } c < r, \\ (\sqrt{c-r} - \sqrt{\gamma})^2 & \text{for } r < c < \gamma + r, \\ 0 & \text{for } c \ge \gamma + r. \end{cases}$$

Note that $\underline{\Lambda}^*$ coincides with the Fenchel-Legendre transform $\underline{\Lambda}^*_{Q^{\eta^*}}$. Theorem 4.1 and Theorem 4.2 thus provide the following explicit

Results for the asymptotic minimization of robust downside risk:

- For any $c \ne r$, $\underline{J}(c) = \inf_{\pi\in\mathcal{A}} \varliminf_{T\uparrow\infty} \frac{1}{T} \ln \sup_{Q^\eta\in\mathcal{Q}} Q^\eta[L_T^\pi \le c] = -\underline{\Lambda}^*(c)$.

- For $c \in (r, \gamma + r)$, the measure Q^{η^*} and the strategy $\widehat{\pi}^c$ defined by

  $$\widehat{\pi}_t^c := \sqrt{\tfrac{c-r}{\gamma}}\,\tfrac{1}{\sigma}(\theta + \eta^{21,*}), \quad t \ge 0,$$

  form a saddle point for the asymptotic minimization of robust downside risk.

- For $c < r$, it is clearly sufficient to invest the whole capital in the bond.

5.2 Geometric Ornstein-Uhlenbeck model with uncertain mean reversion

Consider now a financial market with constant short rate r, where the stock prices are specified by $S_t^1 := \exp(Y_t + \alpha t)$, $\alpha \in \mathbb{R}$. Here the economic factor Y is of Ornstein-Uhlenbeck type, but the investor is uncertain about the "true" future rate of mean reversion.

This situation can be embedded into our general model of Section 2 as follows: Under the reference measure Q_0, the process Y is assumed to be a classical Ornstein-Uhlenbeck process with constant rate of mean reversion $\eta_0 > 0$ and volatility $\sigma > 0$, i. e.,

$$dY_t = -\eta_0 Y_t\, dt + \sigma\, dW_t^1, \quad Y_0 = y_0.$$

Using Itô's formula, it thus follows that

$$dS_t^1 = S_t^1(\alpha\, dt + dY_t + \tfrac{1}{2}\, d\langle Y\rangle_t) = S_t^1((-\eta_0 Y_t + \tfrac{1}{2}\sigma^2 + \alpha)dt + \sigma\, dW_t^1),$$

and so this example corresponds to the choice $g(y) = -\eta_0 y$, $m(y) = -\eta_0 y + \frac{1}{2}\sigma^2 + \alpha$, $\rho_1 = \sigma$, $\rho_2 \equiv 0$, and to the affine market price of risk function

$$\theta(y) = \tfrac{1}{\sigma}(-\eta_0 y + \tfrac{1}{2}\sigma^2 + \alpha - r).$$

To cope with the uncertainty about the "true" future rate of mean reversion, we admit any rate process that is progressively measurable and that takes its values in some interval $[a, b]$, $0 < a \leq \eta_0 \leq b < \infty$. This uncertainty can be embedded into our general model by the ansatz

$$\Gamma = [\tfrac{\eta_0 - b}{\sigma}, \tfrac{\eta_0 - a}{\sigma}] \times \{(0, 0, 0)\}.$$

Indeed, let $Q^\eta \in \mathcal{Q}$ denote the probabilistic model generated by a Γ-valued, progressively measurable process $\eta = (\eta_t)_{t\geq 0}$; cf. (7). In view of (8a), the dynamics of Y under Q^η is given by

$$dY_t = -(\eta_0 - \sigma\eta_t^{11})Y_t\, dt + \sigma\, dW_t^{1,\eta},$$

and the resulting mean reversion process $(\eta_0 - \sigma\eta_t^{11})_{t\geq 0}$ takes values in $[a, b]$.

To determine the dual growth rates $\Lambda(\lambda)$, $\lambda < 0$, explicitly, we now proceed as follows:

Step 1: As in Fleming and Sheu [8] we compute the non-robust growth rates $\Lambda_{Q_0}(\lambda)$ for the reference model Q_0 with constant mean reversion η_0. For this purpose, we apply the results of Subsection 3.3 to the one-point set $\mathcal{Q} = \{Q_0\}$ (i. e. $\Gamma = \{(0, 0, 0, 0)\}$) and use the quadratic ansatz $\varphi(\lambda, y) = \frac{1}{2}Ay^2 + By$. Inserting the derivatives in the EBE (42) and comparing the coefficients of the terms in y^2, in y, and the constants yields a system of equations for A, B and the value $\tilde{\Lambda}_{Q_0}$. The parameter A is determined by a quadratic equation, but one solution is excluded by our regularity assumptions on the solution $(\tilde{\Lambda}_{Q_0}, \varphi(\lambda, \cdot))$. The other root yields the regular pair

$$\tilde{\Lambda}_{Q_0}(\lambda) = \tfrac{1}{2}(1 - \sqrt{1-\lambda})\eta_0 + \lambda\gamma, \quad \varphi(\lambda, y) = \tfrac{1}{2}(1 - \sqrt{1-\lambda})\tfrac{\eta_0}{\sigma^2}y^2 - \tfrac{\lambda}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)y,$$

where $\gamma := r + \frac{1}{2\sigma^2}(\frac{1}{2}\sigma^2 + \alpha - r)^2$. We finally obtain that $\Lambda_{Q_0}(\lambda) = \tilde{\Lambda}_{Q_0}(\lambda)$, in accordance with [8], Theorem 6.1.

Step 2: Since $\Lambda_{Q_0}(\lambda)$ is decreasing in η_0, it is natural to expect that the asymptotic worst-case measure corresponds to the probabilistic model, under which Y has the minimal rate of mean reversion a. To solve the EBE (42) for the robust growth rates $\Lambda(\lambda)$ we thus take the candidate

$$\tilde{\Lambda}(\lambda) = \tfrac{1}{2}(1-\sqrt{1-\lambda})a + \lambda\gamma, \quad \varphi(\lambda, y) = \tfrac{1}{2}(1-\sqrt{1-\lambda})\tfrac{a}{\sigma^2}y^2 - \tfrac{\lambda}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r)y.$$

It is easy to check that this pair indeed solves the EBE (42), and that it satisfies a weak version of our regularity Assumption 3.1; cf. Remark 3.1. This yields the following

Results for the asymptotics of robust expected power utility:

- For any $\lambda < 0$, the optimal growth rate is given by $\Lambda(\lambda) = \frac{1}{2}(1-\sqrt{1-\lambda})a + \lambda\gamma$.
- For any $\lambda \in (-3, 0)$, the trading strategy $\pi_t^*(\lambda) = \pi^*(\lambda, Y_t)$, $t \geq 0$, defined by
$$\pi^*(\lambda, y) = -\tfrac{1}{\sqrt{1-\lambda}}\tfrac{a}{\sigma^2}y + \tfrac{1}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r), \tag{87}$$
is asymptotically optimal for robust expected power utility. For $\lambda \leq -3$, however, our regularity Assumption 3.2 is not satisfied, and the strategy (87) is no longer optimal; see, e. g., [8], Section 6, for a careful discussion.
- The asymptotic worst-case measure $Q^{\eta^*} \in \mathcal{Q}$ does not depend on λ and is determined by the constant control $\eta_t^* = (\frac{\eta_0 - a}{\sigma}, 0, 0, 0)$, $t \geq 0$. Under Q^{η^*} the OU type process Y has the minimal rate of mean reversion a.

The function Λ is continuously differentiable on $(-\infty, 0)$ with $\Lambda'(-\infty) = \gamma$, $\Lambda'(0) = \frac{a}{4} + \gamma$, and the parameter $\lambda[c] < 0$ such that $\Lambda'(\lambda[c]) = c \in (\gamma, \frac{a}{4} + \gamma)$ is given by

$$\lambda[c] = 1 - (\tfrac{a}{4(c-\gamma)})^2.$$

The Fenchel-Legendre transform $\underline{\Lambda}^*$ in (81) thus takes the explicit form

$$\underline{\Lambda}^*(c) = \begin{cases} \infty & \text{for } c \leq \gamma, \\ \frac{(\frac{a}{4} - c + \gamma)^2}{c - \gamma} & \text{for } \gamma < c < \frac{a}{4} + \gamma, \\ 0 & \text{for } c \geq \frac{a}{4} + \gamma. \end{cases}$$

In particular, $\underline{\Lambda}^*$ coincides with $\underline{\Lambda}^*_{Q^{\eta^*}}$. From Theorem 4.2 we deduce the following

Results for the asymptotic minimization of robust downside risk:

- For any $c \neq \gamma$, the optimal rate of decay of robust downside risk $\underline{J}(c)$ coincides with the optimal rate of decay of downside risk $\underline{J}_{Q^{\eta^*}}(c)$ for the model Q^{η^*}, and both rates are given by the Fenchel-Legendre transform $-\underline{\Lambda}^*(c)$.
- For any $c \in (\frac{a}{8} + \gamma, \frac{a}{4} + \gamma)$, the strategy $\widehat{\pi}^c$ defined by
$$\widehat{\pi}_t^c := \pi_t^*(\lambda[c]) = -\tfrac{4}{\sigma^2}(c - \gamma)Y_t + \tfrac{1}{\sigma^2}(\tfrac{1}{2}\sigma^2 + \alpha - r), \quad t \geq 0,$$
yields the optimal rate of decay for robust downside risk and at the same time the optimal rate for the specific model Q^{η^*}, i. e.,
$$\underline{J}(c) = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln \sup_{Q^\eta \in \mathcal{Q}} Q^\eta[L_T^{\widehat{\pi}^c} \leq c] = \lim_{T\uparrow\infty} \tfrac{1}{T} \ln Q^{\eta^*}[L_T^{\widehat{\pi}^c} \leq c] = \underline{J}_{Q^{\eta^*}}(c).$$
- For $c < r$, it is clearly sufficient to invest the whole capital in the bond.

Remark 5.1. *For $c \in (\gamma, \frac{a}{8} + \gamma]$, we cannot establish optimality of $\widehat{\pi}^c := \pi^*(\lambda[c])$ via (85). Indeed, the trading strategy $\pi^*(\lambda)$ is asymptotically optimal for robust expected power utility maximization only if $\lambda \in (-3, 0)$. Since $\lambda[c] \in (-3, 0)$ if and only if $c \in (\frac{a}{8} + \gamma, \frac{a}{4} + \gamma)$, the arguments in (85) are not applicable for $c \in (\gamma, \frac{a}{8} + \gamma]$.*

5.3 Nonlinear coefficients

Let us finally consider an example with nonlinear coefficients. Here the EBE (42) cannot be solved explicitly, but nevertheless the existence of a regular solution can be shown under the following simplifying

Assumption 5.1. *In addition to Assumption 2.1 let us assume that the market price of risk function θ is bounded, that $\Gamma \subset \{(0,0)\} \times \mathbb{R}^2$, and that for all $\lambda < 0$*

$$\exists\, K(\lambda) > 0 : \; g_y(y) + \tfrac{\lambda}{1-\lambda}\rho_1\theta_y(y) \leq -K(\lambda) \quad \textit{for all } y \in \mathbb{R}. \tag{88}$$

Indeed, maximizing among $\nu \in \mathbb{R}$, the EBE (42) takes the condensed form

$$\tilde{\Lambda}(\lambda) = \tfrac{1}{2}\|\rho\|^2\varphi_{yy}(\lambda,y) + \tfrac{1}{2}(\widehat{\rho}(\lambda)\varphi_y(\lambda,y))^2 + \sup_{\eta\in\Gamma}\{n(\lambda,\eta,y) + \varphi_y(\lambda,y)m(\lambda,\eta,y)\}, \tag{89}$$

where we use the notation

$$\begin{aligned}
\widehat{\rho}(\lambda) &:= \sqrt{\tfrac{1}{1-\lambda}\rho_1^2 + \rho_2^2},\\
n(\lambda,\eta,y) &:= \tfrac{1}{2}\tfrac{\lambda}{1-\lambda}[\theta(y) + \eta^{21}]^2 + \lambda r(y) \leq 0,\\
m(\lambda,\eta,y) &:= g(y) + \tfrac{1}{1-\lambda}\rho_1(\lambda\theta(y) + \eta^{21}) + \rho_2\eta^{22}.
\end{aligned}$$

To eliminate the nonlinearity in φ_y, note that

$$\tfrac{1}{2}(\widehat{\rho}(\lambda)\varphi_y(\lambda,y))^2 = \max_{\alpha\in\mathbb{R}}\{\alpha\varphi_y(\lambda,y) - \tfrac{1}{2(\widehat{\rho}(\lambda))^2}\alpha^2\}.$$

Our construction is inspired by arguments in [7], Theorem 7.1. It involves a parameterized family of finite time horizon control problems, where the existence of solutions for the associated HJB equations is already known. The existence of a solution $(\tilde{\Lambda}(\lambda), \varphi(\lambda,\cdot))$ is obtained by taking appropriate limits of these HJB equations.

Proposition 5.1. *Under Assumption 5.1 there exist $\tilde{\Lambda}(\lambda) \in \mathbb{R}_+$, $\varphi(\lambda,\cdot) \in C^2(\mathbb{R})$ that solve the EBE (89). Moreover, the derivative $\varphi_y(\lambda,\cdot)$ is bounded (w. r. t. y), and so this solution also satisfies the regularity Assumptions 3.1 and 3.2. In particular, $\tilde{\Lambda}(\lambda)$ coincides with the growth rate $\Lambda(\lambda)$ of robust expected power utility.*

Proof. Fix $\lambda < 0$ and a stochastic base $(\Omega, \mathcal{G}, \mathbb{G}, R)$ supporting a one-dimensional Brownian motion $(B_t)_{t\geq 0}$. Denote by $\mathcal{Z}_M$ the set of controls α with values in $[-M, M]$, $M \in \mathbb{R}$, and consider the integral criterion

$$\begin{aligned}
V^\tau(y,T) &:= \sup_{\eta\in\mathcal{C}}\sup_{\alpha\in\mathcal{Z}_M} J^\tau(\eta,\alpha,y,T)\\
&:= \sup_{\eta\in\mathcal{C}}\sup_{\alpha\in\mathcal{Z}_M} E\Big[\int_0^T e^{-\tau t}\big(n(\lambda,\eta_t,Y_t^{y,\eta,\alpha}) - \tfrac{1}{2(\widehat{\rho}(\lambda))^2}\alpha_t^2\big)\,dt\Big]
\end{aligned}$$

with discounting rate $\tau > 0$. Here the expectation is taken with respect to R, and for fixed controls $\eta \in \mathcal{C}$, $\alpha \in \mathcal{Z}_M$ the dynamics of $Y^{y,\eta,\alpha}$ follows the SDE

$$dY_t^{y,\eta,\alpha} = [m(\lambda,\eta_t,Y_t^{y,\eta,\alpha}) + \alpha_t]\,dt + \|\rho\|\,dB_t, \quad Y_0^{y,\eta,\alpha} = y.$$

A standard verification argument shows that the value function V^τ is given by the unique classical solution of the HJB equation

$$\tau V + V_T = \tfrac{1}{2}\|\rho\|^2 V_{yy} + \sup_{|\alpha|\leq M}\{\alpha V_y - \tfrac{1}{2(\widehat{\rho}(\lambda))^2}\alpha^2\} + \sup_{\eta\in\Gamma}\{n(\lambda,\eta,\cdot) + V_y m(\lambda,\eta,\cdot)\} \tag{91}$$

with initial condition $V(\cdot,0)\equiv 0$. Here existence and uniqueness of the solution is guaranteed by Theorem IV.4.3 and Remark IV.4.1 in Fleming and Soner [11]. More precisely, the verification result can be obtained in analogy to the proof of Lemma IV.3.1 in [11]; cf. [11], Remark IV.3.3.

In order to construct a solution to the ergodic Bellman equation (89) we are going to analyze the asymptotics of $V^\tau(y,T)$, first for $T\uparrow\infty$ and then for $\tau\downarrow 0$.

1) As a first step we prove the existence of constants $K_1(\lambda), K_2(\lambda)>0$ such that the following estimates are satisfied for all $y\in\mathbb{R}$ and all $T,\tau,M>0$:

$$-K_1(\lambda)\le \tau V^\tau(y,T)\le 0 \tag{92a}$$

$$\tau^{-1}(-K_1(\lambda)-\tfrac{1}{2(\widehat{\rho}(\lambda))^2}M^2)e^{-\tau T}\le V^\tau_T(y,T)\le 0 \tag{92b}$$

$$|V^\tau_y(y,T)|\le \tfrac{K_2(\lambda)}{K(\lambda)} \tag{92c}$$

Since $n\le 0$, the upper bound in (92a) is obvious. With $K_1(\lambda):=\sup_{y\in\mathbb{R},\eta\in\Gamma}|n(\lambda,\eta,y))|$ the lower bound follows from

$$\tau V^\tau(y,T)\ge\tau\sup_{\alpha\in\mathcal{Z}_M}E[\int_0^T e^{-\tau t}(-K_1(\lambda)-\tfrac{1}{2(\widehat{\rho}(\lambda))^2}\alpha_t^2)\,dt]=-K_1(\lambda)\tau\int_0^T e^{-\tau t}\,dt.$$

In order to prove (92b), observe first that $V^\tau(y,T+\delta)\le V^\tau(T,y)$, $T,\delta>0$, due to the fact that $n\le 0$. Thus the map $T\mapsto V^\tau(y,T)$ is decreasing, and this yields the upper bound in (92b). To derive the lower bound, we take $\epsilon,\delta>0$ and choose processes $\widetilde{\eta}\in\mathcal{C}$, $\widetilde{\alpha}\in\mathcal{Z}_M$ such that $V^\tau(y,T)-\epsilon\delta\le J^\tau(\widetilde{\eta},\widetilde{\alpha},y,T)$. This yields

$$\begin{aligned}V^\tau(y,T+\delta)-V^\tau(y,T)+\epsilon\delta&\ge J^\tau(\widetilde{\eta},\widetilde{\alpha},y,T+\delta)-J^\tau(\widetilde{\eta},\widetilde{\alpha},y,T)\\&=E[\int_T^{T+\delta}e^{-\tau t}(n(\lambda,\widetilde{\eta}_t,Y_t^{y,\widetilde{\eta},\widetilde{\alpha}})-\tfrac{1}{2(\widehat{\rho}(\lambda))^2}\widetilde{\alpha}_t^2)\,dt]\\&\ge(-K_1(\lambda)-\tfrac{1}{2(\widehat{\rho}(\lambda))^2}M^2)\int_T^{T+\delta}e^{-\tau t}\,dt.\end{aligned}$$

Dividing by δ and letting ϵ, δ tend to zero we obtain the lower bound in (92b).

It remains to verify (92c). To this end, take $x,y\in\mathbb{R}$ (w. l. o. g. $V^\tau(x,T)>V^\tau(y,T)$) and choose for arbitrary $\epsilon\in(0,V^\tau(x,T)-V^\tau(y,T))$ controls $\widetilde{\eta}\in\mathcal{C}$, $\widetilde{\alpha}\in\mathcal{Z}_M$ such that $V^\tau(x,T)-\epsilon\le J^\tau(\widetilde{\eta},\widetilde{\alpha},x,T)$. Then we get

$$\begin{aligned}|V^\tau(x,T)-V^\tau(y,T)-\epsilon|&\le J^\tau(\widetilde{\eta},\widetilde{\alpha},x,T)-J^\tau(\widetilde{\eta},\widetilde{\alpha},y,T)\\&=E[\int_0^T e^{-\tau t}(n(\lambda,\widetilde{\eta}_t,Y_t^{x,\widetilde{\eta},\widetilde{\alpha}})-n(\lambda,\widetilde{\eta}_t,Y_t^{y,\widetilde{\eta},\widetilde{\alpha}}))\,dt].\end{aligned}$$

Note now that $K_2(\lambda):=\sup_{y\in\mathbb{R},\eta\in\Gamma}|n_y(\lambda,\eta,y)|<\infty$, since the derivatives θ_y, r_y are bounded and Γ is a compact set. The mean value theorem yields

$$n(\lambda,\widetilde{\eta},x)-n(\lambda,\widetilde{\eta},y)\le K_2(\lambda)|x-y|,\quad x,y\in\mathbb{R}.$$

Writing $Y^z:=Y^{z,\widetilde{\eta},\widetilde{\alpha}}$, $z\in\mathbb{R}$, we thus see that

$$\begin{aligned}|V^\tau(x,T)-V^\tau(y,T)-\epsilon|&\le E[\int_0^T e^{-\tau t}K_2(\lambda)|Y_t^x-Y_t^y|\,dt]\\&=K_2(\lambda)\int_0^T e^{-\tau t}E[|Y_t^x-Y_t^y|]\,dt\end{aligned} \tag{93}$$

Let us next derive bounds for $E[|Y_t^x - Y_t^y|]$. Itô's formula applied to $(Y_t^x - Y_t^y)^2$ yields

$$E[(Y_t^x - Y_t^y)^2] = (x-y)^2 + 2E[\int_0^t (Y_s^x - Y_s^y)(m(\lambda, \widetilde{\eta}_s, Y_s^x) - m(\lambda, \widetilde{\eta}_s, Y_s^y))\, ds]. \quad (94)$$

Moreover, the mean value theorem and Assumption 5.1 ensure that

$$(Y_t^x - Y_t^y)(m(\lambda, \widetilde{\eta}_t, Y_t^x) - m(\lambda, \widetilde{\eta}_t, Y_t^y)) \le -K(\lambda)(Y_t^x - Y_t^y)^2.$$

In view of (94) this implies

$$E[(Y_t^x - Y_t^y)^2] \le (x-y)^2 - 2K(\lambda) \int_0^t E[(Y_s^x - Y_s^y)^2]\, ds,$$

and so a Gronwall type argument yields the bound

$$E[|Y_t^x - Y_t^y|]^2 \le E[(Y_t^x - Y_t^y)^2] \le e^{-2K(\lambda)t}|x-y|^2 \quad \text{for all } t \ge 0. \quad (95)$$

Inserting (95) into (93) it now follows that

$$|V^\tau(x,T) - V^\tau(y,T) - \epsilon| \le K_2(\lambda)|x-y| \int_0^T e^{-(\tau + K(\lambda))t}\, dt \le \tfrac{K_2(\lambda)}{\tau + K(\lambda)}|x-y|.$$

Letting ϵ tend to zero in the latter inequality, we conclude that

$$|V_y^\tau(y,T)| = \lim_{x \to y} \tfrac{|V^\tau(x,T) - V^\tau(y,T)|}{|x-y|} \le \tfrac{K_2(\lambda)}{K(\lambda)}.$$

Thus we have shown (92c).

2) Now we are ready to sketch the construction of a solution to (89) by taking the limits $T \uparrow \infty$ and $\tau \downarrow 0$. In view of (92c) there exists M^* such that the solution V^τ of the HJB equation (91) does not depend on M for $M \ge M^*$, i. e., V^τ satisfies

$$\begin{aligned} \tau V^\tau + V_T^\tau &= \tfrac{1}{2}\|\rho\|^2 V_{yy}^\tau + \sup_{\alpha \in \mathbb{R}} \{\alpha V_y^\tau - \tfrac{1}{2(\widehat{\rho}(\lambda))^2}\alpha^2\} + \sup_{\eta \in \Gamma}\{n(\lambda, \eta, \cdot) + V_y^\tau m(\lambda, \eta, \cdot)\} \\ &= \tfrac{1}{2}\|\rho\|^2 V_{yy}^\tau + \tfrac{1}{2}(\widehat{\rho}(\lambda) V_y^\tau)^2 + \sup_{\eta \in \Gamma}\{n(\lambda, \eta, \cdot) + V_y^\tau m(\lambda, \eta, \cdot)\}. \end{aligned} \quad (96)$$

The estimate (92b) ensures that

$$\varphi^\tau(\lambda, y) := \lim_{T \uparrow \infty} V^\tau(y, T)$$

exists pointwise and that $V_T^\tau(y,T)$ vanishes as $T \uparrow \infty$. Standard estimates for PDE's show that $\varphi^\tau(\lambda, \cdot) \in C^2(\mathbb{R})$ and that it satisfies the steady-state form of (96):

$$\tau \varphi^\tau = \tfrac{1}{2}\|\rho\|^2 \varphi_{yy}^\tau + \tfrac{1}{2}(\widehat{\rho}(\lambda)\varphi_y^\tau)^2 + \sup_{\eta \in \Gamma}\{n(\lambda, \eta, \cdot) + \varphi_y^\tau m(\lambda, \eta, \cdot)\}. \quad (97)$$

In particular, $\varphi^\tau(\lambda, \cdot)$ fulfills the steady-state versions of (92a) and (92c). By (92c), the family of functions $\varphi^\tau(\lambda, \cdot) - \varphi^\tau(\lambda, y_0)$ defined for some fixed reference point $y_0 \in \mathbb{R}$, $\tau > 0$, is equicontinuous on compact sets in $\mathbb{R}$. Moreover, $\tau\varphi^\tau(\lambda, y_0)$ is uniformly bounded among all τ, due to (92a). Using the Arzelà-Ascoli theorem, we find a subsequence $\tau_r \to 0$ as $r \uparrow \infty$ such that $\tau_r \varphi^{\tau_r}(\lambda, y_0)$ tends to a limit $\tilde{\Lambda}(\lambda) \le 0$ and such that $\varphi^{\tau_r}(\lambda, \cdot) - \varphi^{\tau_r}(\lambda, y_0)$ converges to a limiting function $\varphi(\lambda, \cdot)$ uniformly on compact sets. In particular, the left-hand side in (97) has the limiting behavior

$$\tau_r \varphi^{\tau_r}(\lambda, y) = \tau_r[\varphi^{\tau_r}(\lambda, y) - \varphi^{\tau_r}(\lambda, y_0)] + \tau_r \varphi^{\tau_r}(\lambda, y_0) \longrightarrow \tilde{\Lambda}(\lambda) \in \mathbb{R}_- \quad \text{as } r \uparrow \infty$$

for all $y \in \mathbb{R}$. Moreover, standard estimates for differential equations show that $\varphi(\lambda, \cdot) \in C^2(\mathbb{R})$ and that the pair $(\tilde{\Lambda}(\lambda), \varphi(\lambda, \cdot))$ indeed satisfies (89). Finally the mean value theorem and the uniform bound $|\varphi_y^\tau(\lambda, y)| \leq K_2(\lambda)/K(\lambda)$ imply

$$|\varphi(\lambda, x) - \varphi(\lambda, y)| = \lim_{\tau_r \downarrow 0} |\varphi^{\tau_r}(\lambda, x) - \varphi^{\tau_r}(\lambda, y)| \leq \tfrac{K_2(\lambda)}{K(\lambda)} |x - y|.$$

Dividing by $|x - y|$ and letting x tend to y shows that $|\varphi_y(\lambda, y)| \leq \frac{K_2(\lambda)}{K(\lambda)}$.

3) It remains to show that the solution $(\tilde{\Lambda}(\lambda), \varphi(\lambda, \cdot))$ is regular in the sense of Assumption 3.1 and 3.2. Since $\varphi_y(\lambda, \cdot)$ is bounded, Assumption 3.1 a) holds obviously and b) follows immediately from (88). Moreover, the function $\varphi(\lambda, \cdot)$ grows at most linearly, i. e., $|\varphi(\lambda, y)| \leq K(1 + |y|)$. This yields the bounds

$$E_{\widehat{R}^\eta}[e^{K(1+|Y_T|)}] \geq E_{\widehat{R}^\eta}[e^{-\varphi(\lambda, Y_T)}] \geq \exp(-K(1 + E_{\widehat{R}^\eta}[|Y_T|])),$$

due to Jensen's inequality. Since b) is satisfied, [24], Lemma 8.2, already shows that $E_{\widehat{R}^\eta}[\exp(K|Y_T|)]$ and $E_{\widehat{R}^\eta}[|Y_T|]$ are bounded uniformly among all processes $\eta \in \mathcal{C}$ and maturities $T \geq 0$. Thus c) follows. Assumption 3.2 is verified by the same arguments. □

6 Outlook

This paper develops a duality approach to the asymptotic minimization of robust downside risk, formulated as a "robust" large deviations control problem. The dual problem corresponds to analyzing the asymptotics of robust expected power utility. Here the solution is related to specific solutions of ergodic Bellman equations. Future research should study the existence and the properties of these solutions more rigorously. As in [29], one should consider instead of L^π more generally $\frac{1}{T} \ln(X_T^\pi / I_T)$ for an index process I of diffusion type. In that case, duality methods to robust utility maximization are no longer applicable to tackle the dual problem. The extended dual problem will lead to a *stochastic differential game* on an infinite time horizon.

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