Liquidity-Adjusted Risk Measures

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Abstract

Liquidity risk is an important type of risk, especially during times of crises. As observed by Acerbi & Scandolo (2008), it requires adjustments to classical portfolio valuation and risk measurement. Main drivers are two dimensions of liquidity risk, namely price impact of trades and limited access to financing. The key contribution of the current paper is the construction of a new, cash-invariant liquidity-adjusted risk measure that can naturally be interpreted as a capital requirement. We clarify the difference between our construction and the one of Acerbi & Scandolo (2008) in the framework of capital requirements using the notion of *eligible assets*, as introduced by Artzner, Delbaen & Koch-Medina (2009). Numerical case studies illustrate how price impact and limited access to financing influence the liquidity-adjusted risk measurements.

Key words: Liquidity, supply-demand curves, risk measures, eligible asset, capital requirements

1 Introduction

Liquidity risk played a major role during many crises that have been observed during the last decades. Its impact was clearly apparent in the recent credit crisis (e.g. the failures of *Bear Stearns* and *Lehman Brothers*), and also during the collapse of *Long Term Capital Management* in 1998. Proper financial regulation and risk management requires appropriate concepts that enable the quantification of liquidity risk. Various aspects of liquidity risk have extensively been investigated during recent years, see e.g. Çetin, Jarrow & Protter (2004), Çetin & Rogers (2007), Jarrow & Protter (2005), Astic & Touzi (2007), and Pennanen & Penner (2010). For further references we refer to a survey article by Schied & Slynko (2011).

A key instrument to control the risk of financial institutions are suitable *measures of the downside risk*. These constitute an important basis for reporting, regulation, and management strategies. The current paper suggests a *liquidity-adjusted measure of the downside risk*, focusing on two key aspects of liquidity risk: access to financing and price impact of trades. These issues receive particular attention in the context of new regulatory standards, such as *Basel III* and *Solvency II*. The proposed liquidity-adjusted risk-measure provides a unified framework beyond the current implementations in practice.

Our approach builds on a recent contribution of Acerbi & Scandolo (2008). While classical asset pricing theory assumes that the value of a portfolio is proportional to the number of its assets, Acerbi & Scandolo (2008) argue that, if access to financing is limited and the size of trades

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impacts prices, the linearity assumption breaks down; classical valuation should be replaced by liquidity-adjusted valuation.

A situation like this occurs, for example, if a fund or bank needs to execute a large block trade. In this case, the realized average price depends on the liquidity of the market as well as the chosen trading strategy. This phenomenon is called price impact. Its magnitude is affected by the specific structure of supply and demand, or, equivalently, the shape of the order book, if the trades are settled on an exchange. The importance of this type of liquidity risk differs among agents and is governed by the particular situation of the investor: adverse price impact is, of course, only relevant, if a particular trade must indeed be quickly executed. Investors with short-term obligations who are subject to strict budget constraints and have no access to cheap external funding might be forced to engage in fire sales. As a consequence, they might tremendously be hurt by price impact; their liquidity risk is large. In contrast, investors with deep pockets will almost not be affected by steep supply-demand curves. They can hold assets over very long time horizons, sell only a few assets simultaneously, and wait until a good price can be realized.

For the convenience of the reader, we review the approach of Acerbi & Scandolo (2008) in Section 2. More specifically, we consider an investor with an asset portfolio in a one-period economy. Limited access to financing is modeled by constraints on borrowing and short selling, or by more general portfolio constraints. At the same time, the investor is faced with temporary short-term obligations which could e.g. be associated with margin calls or withdrawals from customers. In this situation, the investor might be required to liquidate a fraction of her portfolio in order to avoid default. Price impact of orders is explicitly modeled by supply-demand curves. The liquidity-adjusted portfolio value modifies the classical mark-to-market value by accounting for the losses that occur from the forced liquidation of a fraction of the investor's portfolio.

Liquidity-adjusted risk measures can be constructed on the basis of the liquidity-adjusted value. Acerbi & Scandolo (2008) suggested to measure liquidity-adjusted risk by computing a standard monetary risk measure for the liquidity-adjusted value. The resulting liquidity-adjusted risk measure ρ^{AS} is convex, but in general not cash-invariant anymore, and does not possess a natural interpretation as a capital requirement. The key contribution of the current paper is the construction of a new, cash-invariant liquidity-adjusted risk measure ρ^V that can conveniently be interpreted as a capital requirement, see Section 3. Our definition endows ρ^V with a clear operational meaning: it equals the smallest monetary amount that needs to be added to a financial portfolio to make it acceptable. At the same time, ρ^V provides a rationale for convex cash-invariant risk measures, if price impact is important.

Section 3.2 further clarifies the difference between our construction ρ^V and the risk measure ρ^{AS}. For this purpose, we employ the theoretical framework of capital requirements and *eligible assets*, as introduced by Artzner et al. (2009). Section 4 illustrates in the context of numerical case studies how price impact and limited access to financing influence the liquidity-adjusted risk measurements.

2 Liquidity Risk and Portfolio Values

For convenience, the current section recalls the deterministic notion of liquidity-adjusted portfolio value, a concept that was originally proposed by Acerbi & Scandolo (2008). We slightly modify their definition by introducing liquidity constraints that are easily interpretable and by imposing additional portfolio constraints.

2.1 Maximal Mark-to-Market and Liquidation Values

The price of an asset depends on the quantity that is traded. Following e.g. Çetin et al. (2004) and Jarrow & Protter (2005) we capture this fact by supply-demand curves.

Definition 2.1 (Marginal supply-demand curve, best bid, best ask)**.**

(1) Setting $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$, a function $m : \mathbb{R}_* \to \mathbb{R}_+$ is called a *marginal supply-demand curve (MSDC)*, if m is decreasing. We denote by $\mathcal{M}$ the convex cone of all MSDCs.

(2) The numbers $m^+ := m(0+)$ and $m^- := m(0-)$ are called the *best bid* and *best ask*, respectively. Their difference $\Delta m := m^- - m^+ \geq 0$ corresponds to the *bid-ask spread.*

A MSDC m models the current prices of a financial asset or, equivalently, the state of its 'order book' – capturing the dependence of prices on the actual quantities that are traded. If large amounts of an asset are sold, the average price of one unit of the asset will typically be smaller than for small amounts. Conversely, if large amounts are bought, the average price will typically be larger than for small amounts of the asset.

For any number of assets $x \in \mathbb{R}_*$, the price of an infinitesimal additional amount is captured by the marginal price $m(x)$. As a consequence, an investor selling $s \in \mathbb{R}_+$ units of the asset will receive the proceeds

$$\int_0^s m(x)dx.$$

Conversely, if the investor buys $|s| \in \mathbb{R}_+$ units of the asset, she will pay $\int_{-|s|}^0 m(x)dx$, thus 'receive' $\int_0^s m(x)dx \leq 0$.

We endow the the convex cone $\mathcal{M}$ of all MSDCs with a canonical metric for which two MSDCs are close to each other if the corresponding proceeds are close to each other for any number of assets.[1]

A financial market of multiple assets is characterized by a collection of MSDCs.

Definition 2.2 (Spot market, portfolio)**.**

(1) A *spot market* of N risky assets ($N \in \mathbb{N}$) is a vector

$$\bar{m} = (m_0, m_1, \ldots, m_N) \in \mathcal{M}^{N+1}.$$

We will always assume that asset 0 corresponds to cash and set $m_0 \equiv 1$.

(2) A *portfolio* in a spot market of N risky assets is a vector

$$\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N) = (\xi_0, \xi) \in \mathbb{R}^{N+1}$$

whose entries specify the number of assets.

[1]The convex cone $\mathcal{M}$ can be endowed with the metric

$$d_{\mathcal{M}}(m_1, m_2) := |m_1^- - m_2^-| + |m_1^+ - m_2^+| + \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\int_{-n}^{n} |\hat{m}_1(x) - \hat{m}_2(x)| dx \,\wedge 1 \right), \quad m_1, m_2 \in \mathcal{M},$$

where we use the auxiliary function

$$\hat{m}(x) := \begin{cases} m(x) - m^+, & x > 0, \\ 0, & x = 0, \\ m(x) - m^-, & x < 0. \end{cases}$$

In the sequel, all topological properties of $\mathcal{M}$ are based on this metric.

Notation 2.3. For $k \in \mathbb{R}$, $\bar{\xi} = (\xi_0, \xi) \in \mathbb{R}^{N+1}$, we write $k + \bar{\xi} := (\xi_0 + k, \xi)$.

In practice, portfolios are frequently marked-to-market at the best ask and best bid. We will call this value the *maximal mark-to-market.* The maximal mark-to-market is a hypothetical value of a portfolio that cannot always be realized in practice. In fact, unless there are no liquidity effects, the mark-to-market value typically differs from the *liquidation value*, i.e. from the income or cost of an immediate liquidation of the portfolio. The liquidation value does not only depend on the best bid and best ask, but on the whole supply-demand curve.

Definition 2.4 (Def. 4.6 & 4.7 in Acerbi & Scandolo (2008))**.** Let $\bar{\xi} \in \mathbb{R}^{N+1}$ be a portfolio in a spot market of N risky assets.

(1) The *liquidation value* of $\bar{\xi}$ is given by

$$L(\bar{\xi}, \bar{m}) \;:=\; \sum_{i=0}^{N} \int_0^{\xi_i} m_i(x)dx \;=\; \xi_0 + \sum_{i=1}^{N} \int_0^{\xi_i} m_i(x)dx.$$

(2) The *maximal mark-to-market value* of a portfolio $\bar{\xi}$ is given by

$$U(\bar{\xi}, \bar{m}) \;:=\; \xi_0 + \sum_{i=1}^{N} m_i^{\pm}(\xi_i) \cdot \xi_i,$$

where $m_i^{\pm}(\xi_i) = \begin{cases} m_i^{+}, & \text{if } \xi_i \geq 0, \\ m_i^{-}, & \text{if } \xi_i < 0. \end{cases}$

Remark 2.5. Acerbi & Scandolo (2008) call the function U the *uppermost mark-to-market value.*

The following remark summarizes useful properties of L and U.

Remark 2.6 (Properties, see Section 4 in Acerbi & Scandolo (2008))**.**

(1) L and U are continuous on $\mathbb{R}^{N+1} \times \mathcal{M}^{N+1}$.

(2) L and U are concave functions of their first argument (i.e. the portfolio) that are differentiable on $\mathbb{R} \times \mathbb{R}_*^N$.

(3) Let $\bar{\xi} \in \mathbb{R}^{N+1}$ be some portfolio and $\bar{m} \in \mathcal{M}^{N+1}$ a spot market.
- If $\lambda \geq 1$, then $L(\lambda\bar{\xi}, \bar{m}) \leq \lambda L(\bar{\xi}, \bar{m})$. If $0 \leq \lambda \leq 1$, then $L(\lambda\bar{\xi}, \bar{m}) \geq \lambda L(\bar{\xi}, \bar{m})$.
- U is positively homogeneous, i.e. for $\lambda \geq 0$ we have that $U(\lambda\bar{\xi}, \bar{m}) = \lambda U(\bar{\xi}, \bar{m})$.

(4) U and L are fully decomposable.[2]

(5) $\forall k \in \mathbb{R}$: $U(k + \bar{\xi}, \bar{m}) = k + U(\bar{\xi}, \bar{m})$, $L(k + \bar{\xi}, \bar{m}) = k + L(\bar{\xi}, \bar{m})$.

An investor can buy and sell assets and thereby change her portfolio at prevailing market prices. Letting $\bar{\xi} \in \mathbb{R}^{N+1}$ be a portfolio, an investor can liquidate a subportfolio $(0, \gamma)$, $\gamma \in \mathbb{R}^N$, changing the cash position of the portfolio by the liquidation value $L((0, \gamma), \bar{m})$ of the subportfolio $(0, \gamma)$. Any portfolio which is attainable from $\bar{\xi}$ thus has the form:

$$\left(\xi_0 + \sum_{i=1}^{N} \int_0^{\gamma_i} m_i(x)dx, \xi - \gamma\right) \quad (\gamma \in \mathbb{R}^N).$$

[2] A function $f : \mathbb{R}^{N+1} \to \mathbb{R}$ is *fully decomposable*, if $f(x_0, x_1, \ldots, x_N) = \sum_{i=0}^{N} f_i(x_i)$ for functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 0, 1, \ldots, N$.

Definition 2.7. We denote by $\mathcal{A}(\bar{\xi}, \bar{m})$ the set of all portfolios which are attainable from $\bar{\xi}$ in the spot market $\bar{m}$.

2.2 Liquidity and portfolio constraints

Classical portfolio theory assumes that the value of a portfolio does not depend on its owner. The value is a linear function of the number of assets. Acerbi & Scandolo (2008) argue convincingly that this standard approach is not correct if prices depend on the quantities traded and if investors have at the same time limited access to financing:

Investors typically need to fulfill short-term obligations, but cannot always quickly borrow liquidity on financial markets. If short of cash, they need to liquidate a part of their portfolio. The average prices, however, at which investors can sell (or buy) assets depend in the presence of price impact on the quantities that are traded. In this sense, the portfolio value depends on the specific financial situation of the investor as well as on the supply-demand curves of the assets.

In order to model these effects, we will characterize the investor by two different constraints that can be observed in real markets: *liquidity constraints* and *portfolio constraints*. Liquidity constraints signify short-time payments an investor needs to make. Portfolio constraints refer, for example, to borrowing and short selling constraints.

Liquidity constraints We consider a one period economy with time points $t = 0, 1$. An owner of an asset portfolio will typically receive certain payments, or is required to fulfill certain financial obligations – including, for example, items like rent payments, maintenance costs, coupons, or margin payments. The total amount of these cash flows will affect the cash position of the investor. For modeling purposes, we will assume that payments occur at the end of the time horizon, i.e. at $t = 1$, and are given by a function of the assets other than cash that the investor holds at time 1.

Definition 2.8. The *short-term cash flows* (SCF) are a function $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ such that $\phi(0_N) = 0$. We will write $\phi \in \mathrm{SCF}$.

An investor is required to be sufficiently liquid at the end of the time horizon $t = 1$: she must own enough cash to cover any obligations due. Otherwise, the investor will default. Typically, an investor has a borrowing constraint that prevents her from obtaining an arbitrarily large amount of cash. The difference between liquid and illiquid portfolios is captured by the following definition.

Definition 2.9. Let $\phi \in \mathrm{SCF}$, and $a \in \mathbb{R}$. The set of *liquid portfolios* which are attainable from $\bar{\xi} \in \mathbb{R}^{N+1}$ is defined by

$$\mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) = \left\{ \bar{\eta} \in \mathcal{A}(\bar{\xi}, \bar{m}) : \eta_0 + \phi(\eta) \geq a \right\}.$$

The pair (ϕ, a) is called a *liquidity constraint.*

The number a is typically negative and signifies the maximal amount the investor can borrow. We assume in this case that the investor is prohibited to borrow more than $|a|$. $\phi(\eta)$ signifies the short-term cash flows associated with a portfolio $\bar{\eta} \in \mathcal{A}(\bar{\xi}, \bar{m})$. These cash flows plus available cash must exceed a.

Remark 2.10. Def. 2.9 assumes that short-term cash flows are not directly determined from the original portfolio $\bar{\xi}$, but from liquid portfolios that can be attained from $\bar{\xi}$. Alternatively, one could, of course, assume that short-term cash flows are associated with the original portfolio and modify the theory accordingly. The economic interpretations of these two conceivable alternatives differ slightly:
Our convention essentially assumes that short-term cash flows are due at the beginning of a time period immediately after the investor decides about the composition of the portfolio (that needs to satisfy the constraints). Alternatively, one could assume that short-term cash flows are due at the end of a time period.

Stylized examples of liquidity constraints are proportional margin constraints and convex constraints.

Example 2.11. (1) *Proportional margin constraints*: the obligations due to holding the assets other than cash are proportional to the number of assets on which the investor is short, i.e.

$$\phi(\eta) = -\sum_{i=1}^{N} \alpha_i \cdot \eta_i^-, \quad \alpha_i \geq 0, \quad i = 1, \ldots, N$$

(2) *Convex constraints*: $\phi \leq 0$ is a concave function with $\phi(0_N) = 0$. Proportional margin constraints are a special case of convex constraints.

Remark 2.12. Acerbi & Scandolo (2008) introduce the concept of a "liquidity policy" which is a convex and closed subset $\mathcal{C} \subseteq \mathbb{R}^{N+1}$ such that

(1) $\bar{\eta} \in \mathcal{C} \quad \Rightarrow \quad \bar{\eta} + b := \bar{\eta} + (b, 0_N) \in \mathcal{C} \;\; \forall b > 0$

(2) $\bar{\eta} = (\eta_0, \eta) \in \mathcal{C} \quad \Rightarrow \quad (\eta_0, 0_N) \in \mathcal{C}$

The constraints in Example 2.11 correspond to special cases of liquidity policies. Conversely, if $\inf\{\eta_0 : (\eta_0, \eta) \in \mathcal{C}\} > -\infty$ for all $\eta \in \mathbb{R}^N$, a liquidity policy $\mathcal{C}$ induces a liquidity constraint by setting

$$\begin{aligned} a &= \inf\{\eta_0 : (\eta_0, 0_N) \in \mathcal{C}\}, \\ \phi(\eta) &= -\inf\{\eta_0 : (\eta_0, \eta) \in \mathcal{C}\} + a, \end{aligned}$$

with the usual convention that $\inf \emptyset = \infty$.

Portfolio constraints Real investors are also restricted by other constraints that limit the feasibility of trading strategies. These portfolio constraints are typically formulated in terms of a non-empty, closed, convex set $\mathcal{K} \subseteq \mathbb{R}^N$. It is required that $\eta \in \mathcal{K}$ for any admissible portfolio $\bar{\eta} = (\eta_0, \eta)$ at the end of the time horizon, $t = 1$. We suppose that $0_N \in \mathcal{K}$, i.e. holding cash only is acceptable, as long as the borrowing constraint $\eta_0 \geq a$ is satisfied.

Example 2.13. (1) *Unconstrained case*: $\mathcal{K} = \mathbb{R}^N$

(2) *Constraints on short-selling*: $\mathcal{K} = [-q_1, \infty) \times [-q_2, \infty) \times \cdots \times [-q_N, \infty)$ for $q_i \geq 0$, $i = 1, \ldots, N$

(3) *Cone constraints*: $\mathcal{K}$ is a non-empty, closed, convex cone in $\mathbb{R}^N$.

2.3 The value of a portfolio

An investor who owns a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ at time $t = 0$ might need to liquidate a fraction of her portfolio in order to meet the liquidity and portfolio constraints at time $t = 1$: short-term payments need to be made, but borrowing and short selling are typically restricted. The liquidation of assets will, however, typically not occur at the best bid, unless supply-demand curves are horizontal. The supply-demand curve determines the proceeds of any transaction, and both average and marginal prices are functions of the number of assets that are traded. The liquidity-adjusted value that we define in this section takes these issues into account. Our definition of the portfolio value follows conceptually the ideas of Acerbi & Scandolo (2008). Liquidity-adjusted risk measures are, however, defined differently, see Section 3.1.

Although often used in practice, the maximal mark-to-market value is an artificial quantity. Measuring the value of a portfolio by its maximal mark-to-market has at least one important disadvantage: liquidity effects are completely neglected. If supply-demand curves were horizontal, then the maximal mark-to-market value could indeed be interpreted as the value of a portfolio. In reality, however, supply-demand curves are typically not horizontal which complicates the situation significantly. When liquidity and portfolio constraints are absent, the maximal mark-to-market value can be interpreted as a market-based approximation to the long-run value of a portfolio. If, however, short-term obligations and portfolio constraints are present – possibly forcing investors to liquidate a fraction of their assets, the maximal mark-to-market value becomes an inadequate approximation of the portfolio value.

The approach that we follow in this paper requires that the mark-to-market value must only be used as an approximation of the portfolio value if a portfolio satisfies all liquidity and portfolio constraints. If a portfolio does *not* satisfy these constraints, we *require* that a suitable fraction of the portfolio is liquidated, *before* the mark-to-market value is computed. This procedure – originally suggested by Acerbi & Scandolo (2008) – thereby assigns a cost to illiquidity. The value of a portfolio is then given by the maximal mark-to-market value, after a suitable part of the original portfolio has been liquidated.

For the formal definition, we consider again an economy with two dates $t = 0, 1$.

Definition 2.14. The value of a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ under the liquidity constraint (ϕ, a) and the portfolio constraint $\mathcal{K}$ is given by

$$V(\bar{\xi}, \bar{m}) = V(\bar{\xi}, \bar{m}, \phi, a, \mathcal{K}) = \sup\{U(\bar{\eta}, \bar{m}) : \bar{\eta} \in \mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})\}. \tag{1}$$

Remark 2.15. If ϕ is concave, the valuation problem amounts to maximizing the concave function $U(\cdot, \bar{m})$ on the convex set of attainable liquid portfolios $\mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$. If ϕ models obligations of the investor (which is typically the most interesting case), then ϕ will be non positive.

Assumption 2.16. From now on we will always assume that the SCF ϕ are concave and non positive. This will be captured by the following definition.

Definition 2.17. We denote by Φ the family of all concave and non positive functions on $\mathbb{R}^N$. We endow Φ with the uniform distance d_∞, i.e. if $\phi, \psi \in \Phi$, then $d_\infty(\phi, \psi) = \sup_{x \in \mathbb{R}^N} |\phi(x) - \psi(x)|$. Φ is called the *family of concave short-term cash flows.*

Proposition 2.18. *Suppose that Assumption 2.16 holds. Then the value map V has the following properties:*

(1) *The maximal mark-to-market dominates the value map:*

$$V(\bar{\xi},\bar{m}) \le U(\bar{\xi},\bar{m}).$$

This implies, in particular, that $V(\bar{\xi},\bar{m}) < \infty$.

(2) *Suppose that* $L(\bar{\xi},\bar{m}) \ge a$. *Then* $V(\bar{\xi},\bar{m}) > -\infty$ *(or, equivalently,* $\mathcal{L}(\bar{\xi},\bar{m},\phi,a)\cap(\mathbb{R}\times\mathcal{K}) \neq \emptyset$*), and* $V(\bar{\xi},\bar{m}) \ge L(\bar{\xi},\bar{m})$.

(3) *Concavity: for* $\alpha \in [0,1]$ *and* $\bar{\xi}^1, \bar{\xi}^2 \in \mathbb{R}^{N+1}$ *we have*

$$V(\alpha\bar{\xi}^1 + (1-\alpha)\bar{\xi}^2, \bar{m}) \ge \alpha V(\bar{\xi}^1,\bar{m}) + (1-\alpha)V(\bar{\xi}^2,\bar{m}).$$

(4) *Translation-supervariance: for all* $k \ge 0$ *and* $\bar{\xi} \in \mathbb{R}^{N+1}$ *we have*

$$V(\bar{\xi}+k,\bar{m}) \ge V(\bar{\xi},\bar{m}) + k. \tag{2}$$

(5) *Monotonicity: if* $\bar{\xi} \le \bar{\eta}$, *then* $V(\bar{\xi},\bar{m}) \le V(\bar{\eta},\bar{m})$.

Proof. See Section A. □

Remark 2.19. Suppose that the portfolio constraint $\mathcal{K}$ can be expressed in terms of r convex functions $\psi_1,\dots,\psi_r : \mathbb{R}^N \to \mathbb{R}$, i.e.

$$\eta \in \mathcal{K} \qquad \Longleftrightarrow \qquad \psi_1(\eta) \le 0, \dots, \psi_r(\eta) \le 0.$$

This condition is obviously satisfied for the cases presented in Example 2.13. In this situation, the portfolio value (1) can be characterized via Lagrange multipliers. Indeed, a portfolio $\bar{\eta}$ is attainable from $\bar{\xi}$, if

$$\eta_0 - \xi_0 - \sum_{i=1}^{N} \int_0^{\xi_i-\eta_i} m_i(x)dx = 0.$$

This constraint can be replaced by an inequality constraint that does not affect the value in (1). The objective is thus to determine the supremum of $U(\bar{\eta},\bar{m})$ for varying $\bar{\eta}$ under the following $r+2$ inequality constraints:

(1) *Attainability*: $\nu_1(\bar{\eta}) := \eta_0 - \xi_0 - \sum_{i=1}^{N} \int_0^{\xi_i-\eta_i} m_i(x)dx \le 0$

(2) *Liquidity constraint*: $\nu_2(\bar{\eta}) := a - \eta_0 - \phi(\eta) \le 0$

(3) *Portfolio constraints*: $\psi_1(\eta) \le 0, \dots, \psi_r(\eta) \le 0$

This is a standard optimization problem, see, e.g., Section 28 in Rockafellar (1996). If the short-term cash flows and the portfolio constraints are fully decomposable, the optimization problem, can be reduced to $N+1$ one-dimensional unconstrained optimization problems and the determination of a Karush-Kuhn-Tucker vector. The assumption of decomposability greatly simplifies the analysis and is not too unrealistic to capture examples in practice.

3 Liquidity Risk and Risk Measures

Definition (1) of the liquidity-adjusted value of a portfolio does not yet involve any randomness. So far, the portfolio value is a deterministic function of both the deterministic supply-demand curves and the deterministic short-term cash flows. We will now assume that at least one of these quantities is not revealed when portfolio risk is measured.

Consider again an economy with two dates $t = 0, 1$. The portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ whose risk needs to be measured is given at time $t = 0$. Again, liquidity and portfolio constraints are imposed at time 1. These do possibly require that a suitable fraction of the portfolio is liquidated. At time $t = 0$ the constraints are typically not completely known, since supply-demand curves and, possibly, short-term cash flows are random quantities that are not revealed until time $t = 1$. As a consequence, no action needs to be taken by the investor until time 1. At time 1, however, liquidity and portfolio constraints need to be respected, once the realizations of supply-demand curves and short-term cash flows are known. Pathwise the liquidity-adjusted value of the portfolio is then again defined by (1), but becomes now a random variable that is measurable with respect to the information that is available at time 1.

The goal of the current section is to define a liquidity-adjusted risk of the portfolio. The sought risk measure will be given as a *capital requirement* in Section 3.1, i.e. the smallest monetary amount that needs to be added to the portfolio at time 0 to make it acceptable. This approach ensures that the liquidity-adjusted risk measure is indeed a generalized convex monetary risk measure on the set of portfolios.

Section 3.2 illuminates the difference between our approach and the approach of Acerbi & Scandolo (2008) on liquidity-adjusted risk measure within a general framework of risk measures associated to capital requirements, see Artzner et al. (2009). The key concept is the notion of an *eligibile asset* which provides the reference point with respect to which capital requirements are computed.

3.1 Liquidity-adjusted risk

Impact of randomness Let $(\Omega, \mathcal{F})$ denote a measurable space which models uncertainty, and let P be a given probability measure on $(\Omega, \mathcal{F})$. We will endow the metric spaces (Φ, d_∞) and $(\mathcal{M}, d_{\mathcal{M}})$ with the corresponding Borel-σ-algebras. Note that Φ and $\mathcal{M}$ are thus Standard-Borel spaces.

Definition 3.1 (Random MSDC, random SCF)**.**

(1) A *random marginal supply-demand curve* (MSDC) is a vector $\bar{m} = (1, m_1, \ldots, m_N)$ of measurable mappings $m_i : \Omega \to \mathcal{M}$, $i = 1, \ldots, N$. The vector $\bar{m} = (1, m_1, \ldots, m_N)$ of random MSDCs corresponds to a *random spot market* of N risky assets.

(2) A *random short-term cash flow* (SCF) is a measurable mapping $\phi : \Omega \to \Phi$.

If MSDCs and SCFs are random, then the liquidity-adjusted value of a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ as defined in (1) is a random variable that needs to be computed for almost all scenarios $\omega \in \Omega$.

Definition 3.2. Let $\bar{m}$ be a MSDC, ϕ a SCF, $a \in \mathbb{R}$, and $\mathcal{K}$ a portfolio constraint. The *random (liquidity-adjusted) value* of the portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ is defined by

$$\Omega \to \mathbb{R} \cup \{-\infty\}, \quad \omega \mapsto V(\bar{\xi}, \bar{m}(\omega), \phi(\omega), a, \mathcal{K}).$$

We will sometimes simply write $V(\bar{\xi})$ for the random value of $\bar{\xi}$.

Our goal is to measure the risk of a portfolio $\bar{\xi}$ in terms of random values of cash-adjusted portfolios. The following assumption ensures that $L(\bar{\xi}, \bar{m}), U(\bar{\xi}, \bar{m})$ belong to $L^\infty := L^\infty(\Omega, \mathcal{F}, P)$ for all $\bar{\xi} \in \mathbb{R}^{N+1}$.

Assumption 3.3. For all $i = 1, \dots, N$ and $x \in \mathbb{R}_*$: $m_i(x) \in L^\infty$.

Liquidity-adjusted risk measure For the definition of liquidity-adjusted risk measures, we fix a convex risk measure ρ on L^∞, as described in Section 4.3 in Föllmer & Schied (2011). It is well-known that ρ induces an acceptance set $\mathcal{A}$ from which it can be recovered as a capital requirement, see, e.g., Section 4.1 in Föllmer & Schied (2011).

Acerbi & Scandolo (2008) suggest measuring the liquidity-adjusted risk of a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ by

$$\rho^{\mathrm{AS}}(\bar{\xi}) := \rho(V(\bar{\xi})),$$

i.e. by applying a classical risk measure to the liquidity-adjusted value. We propose an alternative definition of a liquidity-adjusted risk measure which is based on the notion of capital requirements. In contrast to the liquidity-adjusted risk measure of Acerbi & Scandolo (2008) our liquidity-adjusted risk measure remains cash-invariant and thus measures risk on a monetary scale.

Definition 3.4. The liquidity-adjusted risk of a portfolio $\bar{\xi}$ is defined as

$$\rho^V(\bar{\xi}) := \inf\{k \in \mathbb{R} : V(k + \bar{\xi}) \in \mathcal{A}\}.$$

Def. 3.4 defines liquidity-adjusted risk as the smallest monetary amount that has to be added at time 0 such that its liquidity-adjusted random value at time 1 is acceptable for the risk measure ρ. Since the liquidity-adjusted value incorporates price effects as well as the size of short-term cash flows and access to financing, the liquidity-adjusted risk measure will quantify these influences. We will illustrate this in the context of a numerical case study in Section 4 and provide comparative statics.

It is easy to see that for a given risk measure ρ our liquidity-adjusted risk and the one suggested by Acerbi & Scandolo (2008) have the same sign; however, the absolute value of ρ^{AS} is always larger than the one of ρ^V.

Proposition 3.5. *If $\bar{\xi} \in \mathbb{R}^{N+1}$ is a portfolio, then $\rho^V(\bar{\xi}) \in \mathbb{R}$. Moreover,*

$$|\rho^V(\bar{\xi})| \leq |\rho^{\mathrm{AS}}(\bar{\xi})|, \tag{3}$$

and $\rho^V(\bar{\xi})$ and $\rho^{\mathrm{AS}}(\bar{\xi})$ have the same sign, if $\rho^V(\bar{\xi}) \neq 0$.

Proof. See Section A. □

The mapping ρ^V defines a liquidity-adjusted risk measure that is cash-invariant and that can be interpreted as a capital requirement.

Theorem 3.6. *The mapping $\rho^V : \mathbb{R}^{N+1} \to \mathbb{R}$ is inverse monotone and convex as well as cash-invariant in the following sense:*

$$\rho^V(\bar{\xi} + k) = \rho^V(\bar{\xi}) - k \quad \text{for all } k \in \mathbb{R}.$$

The acceptance set

$$\mathcal{A}^V := \{\bar{\xi} \in \mathbb{R}^{N+1} : V(\bar{\xi}) \in \mathcal{A}\}$$

is convex, and ρ^V can be recovered from $\mathcal{A}^V$ as a capital requirement:

$$\rho^V(\bar{\xi}) = \inf\{k \in \mathbb{R} : \bar{\xi} + k \in \mathcal{A}^V\}.$$

The mapping ρ^V is called liquidity-adjusted risk measure*.*

Proof. See Section A. □

For a numerical implementation of liquidity-adjusted risk measures the following implicit equation is useful.

Theorem 3.7. *Let ρ be a convex risk measure that is continuous from above, and assume that P-almost surely $\bar{\eta} \mapsto V(\bar{\eta})$ is continuous on the interior of its essential domain. Suppose that $\bar{\xi}$ is a portfolio such that $\bar{\xi} + \rho^V(\bar{\xi})$ is P-almost surely in the interior of the essential domain of V. Then the liquidity-adjusted risk $\rho^V(\bar{\xi})$ is equal to the unique solution $k \in \mathbb{R}$ of the equation*

$$0 = \rho(V(\bar{\xi} + k)). \tag{4}$$

Proof. See Section A. □

3.2 Liquidity-adjusted risk measures and capital requirements

The definitions of our liquidity-adjusted risk measure and the one introduced by Acerbi & Scandolo (2008) can be embedded into a conceptual framework that was provided by Artzner et al. (2009). Their paper describes the process of measuring risk as follows:

> Measuring the risk of a portfolio of assets and liabilities by determining the minimum amount of capital that needs to be added to the portfolio to make the future value "acceptable" has now become a standard in the financial service industry. [...]
>
> [This approach requires to specify] a traded asset in which the supporting capital may be invested (the "eligible asset" [...]). [...]
>
> The minimum required capital will of course depend on the definition of acceptability, but also on the choice of the eligible asset.

The notion of capital requirements and *eligible assets* facilitates a comparison between ρ^V and ρ^{AS}. As suggested in Artzner et al. (2009), we specify a set of acceptable positions at the end of the time horizon and then compare different eligible assets.[3] For a given eligible asset, the implied liquidity-adjusted risk measure is the smallest number of shares of the eligible asset that need to be added to the portfolio to make its liquidity-adjusted value acceptable.

Suppose that we are in the situation described in the previous Section 3.1. For the purpose of formally defining liquidity-adjusted capital requirements relative to an eligible asset, we enlarge the initial spot market $\bar{m}$ by one further asset $e \in \mathcal{M}$ to be interpreted as the eligible asset. The extended spot market is thus given by the vector $\tilde{m} = (\bar{m}, e) \in \mathcal{M}^{N+2}$, an extended portfolio is a vector $\tilde{\xi} = (\bar{\xi}, k) \in \mathbb{R}^{N+2}$. We do not impose any further restrictions on the eligible asset e, i.e. we allow, for example, redundancy if $e \equiv 1$.

[3] An alternative approach, suggested by Filipović (2008), investigates the effect of a change of numeraire on risk measures. In this case, the acceptance set of nominal final values is dependent on the numeraire. For our comparison between ρ^V and ρ^{AS} the closely related framework of Artzner et al. (2009) is preferable since it fixes an acceptance set independently of the eligible asset.

Risk of an original portfolio $\bar{\xi}$ is measured in units of the eligible asset. For this reason, the eligible asset should not distort the original liquidity constraints. This can be formalized by assuming that the short-term cash flows in the extended market are given by

$$\tilde{\phi}(\omega)((\eta, k)) := \phi(\omega)(\eta) \quad (\omega \in \Omega, \eta \in \mathbb{R}^N, k \in \mathbb{R}).$$

Objects associated with the extended market will be marked by a tilde, i.e. $\tilde{m}$, $\tilde{L}$, $\tilde{U}$, $\tilde{\mathcal{L}}$, $\tilde{\phi}$, $\tilde{\mathcal{K}}$, $\tilde{V}$, in order to differentiate between the original and the extended market.

Definition 3.8. Liquidity-adjusted risk of a portfolio $\bar{\xi}$ *relative to the eligible asset* e is defined as the smallest number of assets that need to be added to a portfolio to make its liquidity-adjusted value acceptable, i.e.

$$\rho^e(\bar{\xi}) := \inf\{k \in \mathbb{R} : \tilde{V}((\bar{\xi}, k)) \in \mathcal{A}\}. \tag{5}$$

The following proposition is an immediate consequence of the definitions and an appropriately modified proof of Theorem 3.6.

Proposition 3.9. *The mapping $\rho^e : \mathbb{R}^{N+1} \to \mathbb{R}$ has the following properties:*

- Inverse monotonicity*: if $\bar{\xi} \leq \bar{\eta}$, then $\rho^e(\bar{\xi}) \geq \rho^e(\bar{\eta})$.*
- Convexity: *for all $\alpha \in [0, 1]$ and $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{N+1}$ we have*

$$\rho^e(\alpha\bar{\xi} + (1 - \alpha)\bar{\eta}) \leq \alpha\rho^e(\bar{\xi}) + (1 - \alpha)\rho^e(\bar{\eta}).$$

Both liquidity-adjusted risk measures ρ^V and ρ^{AS} can be recovered from ρ^e for suitably chosen e:

(1) ρ^V corresponds to the special case $e \equiv 1$ and the choice $\tilde{\mathcal{K}} = \mathcal{K} \times \mathbb{R}_+$. This is apparent from the definitions. ρ^V can thus be interpreted as the smallest monetary amount that needs to be added to the portfolio at time 0 to make its liquidity-adjusted value acceptable.

(2) ρ^{AS} is not a special case of $\tilde{\rho}^e$ for an appropriate e, but a limiting case for a suitably chosen sequence of eligible assets.

To state this more precisely, we consider the portfolio constraint $\tilde{\mathcal{K}} = \mathcal{K} \times \mathbb{R}$ and the family

$$e_\epsilon(x) := \begin{cases} 0, & \text{if } x > \epsilon, \\ 1, & \text{if } -\epsilon \leq x \leq \epsilon, \\ 2, & \text{if } x < -\epsilon, \end{cases} \qquad (\epsilon > 0)$$

of "random" MSDCs for the eligible asset. This choice signifies that selling more than ϵ units of the eligible asset as well as buying more than ϵ units is suboptimal. In the limiting case $\epsilon \downarrow 0$, the eligible asset becomes completely illiquid. The limiting case formalizes in the context of Artzner et al. (2009) that ρ^{AS} is the smallest monetary amount that needs to be added to the liquidity-adjusted value at time 1 to make it acceptable.

Proposition 3.10. *Suppose that P-almost surely $\bar{\eta} \to V(\bar{\eta})$ is continuous on the interior of its essential domain and that the reference risk measure ρ is continuous from above and below. Let ρ^{e_ϵ} denote the capital requirement (5) relative to the eligible asset e_ϵ. Suppose that $\bar{\xi} \in \mathbb{R}^{N+1}$ is P-almost surely in the interior of the essential domain of V. Then we have*

$$\rho^{\mathrm{AS}}(\bar{\xi}) = \lim_{\epsilon \downarrow 0} \rho^{e_\epsilon}(\bar{\xi}).$$

Proof. See Section A. □

Remark 3.11. Acerbi & Scandolo (2008) argue that risk measures should be coherent, if no liquidity risk is present. Liquidity-adjusted risk measures should thus be coherent in the extreme case of a spot market in which all MSDCs are horizontal.

Consider again the setup of Section 3.1 – now *assuming* that ρ is coherent. The liquidity-adjusted risk measure ρ^V will indeed be coherent, if the MSDCs in the spot market are horizontal. In the case of general MSDCs, ρ^V will be convex and cash-invariant. Example 3.12 illustrates that ρ^V will typically not anymore be positively homogeneous, if MSDCs are not horizontal. – Conceptually, the risk measure ρ^V provides *a rationale for convex risk measures*, if price impact is important. At the same time, ρ^V measures risk as the minimal cash amount that makes the future value of the position acceptable.

In contrast to ρ^V, the liquidity-adjusted risk measure ρ^{AS} does not preserve cash-invariance. In the extreme case of horizontal MSDCs, ρ^{AS} is also coherent. If MSDCs are not horizontal, the risk measure ρ^{AS} of Acerbi & Scandolo (2008) is convex, but *not* cash-invariant.

Example 3.12. Consider a spot market with only one risky asset whose MSDC is given by

$$m_1(x) := \begin{cases} 1 - x, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Suppose furthermore that there are no portfolio constraints and that the liquidity constraint is given by $\phi(\eta_1) = -|\eta_1|$ and $a = 0$. The coherent risk measure ρ is given by minus the expectation operator.

In this framework, the portfolio $\bar{\xi} = (0, 1)$ satisfies $\rho^V(\bar{\xi}) = -0.5$. If ρ^V would be positively homogeneous, then $\rho^V(2\bar{\xi}) = -1$. However, adding the capital injection of $-0.9 > -1 = 2\rho^V(\bar{\xi})$ to the scaled portfolio $2\bar{\xi}$ yields the portfolio value $V((-0.9, 2)) = -\infty < 0$, hence $\rho^V(2\bar{\xi}) > 2\rho^V(\bar{\xi})$.

4 Numerical case study

Equation (4) provides a convenient characterization of liquidity-adjusted risk that we will now exploit for the specific reference risk measure *utility-based shortfall risk*, see Weber (2006), Giesecke, Schmidt & Weber (2008), and Föllmer & Schied (2011). In this setting, we will see how the various ingredients of the framework, such as the MSDC, liquidity and portfolio constraints, affect the liquidity-adjusted risk of a portfolio. For comparison, we will also investigate the risk measures value at risk and average value at risk.

Definition 4.1. For a given convex loss function[4] ℓ and an interior point z in the range of ℓ, we define a convex acceptance set by

$$\mathcal{A} := \{X \in L^\infty : E[\ell(-X)] \leq z\}.$$

The risk measure $\rho : L^\infty \to \mathbb{R}$ defined by

$$\rho(X) := \inf\{k \in \mathbb{R} : X + k \in \mathcal{A}\}$$

is called *utility-based shortfall risk* (UBSR).

[4] An increasing, non constant function $\ell : \mathbb{R} \to \mathbb{R}$ is called a *loss function*.

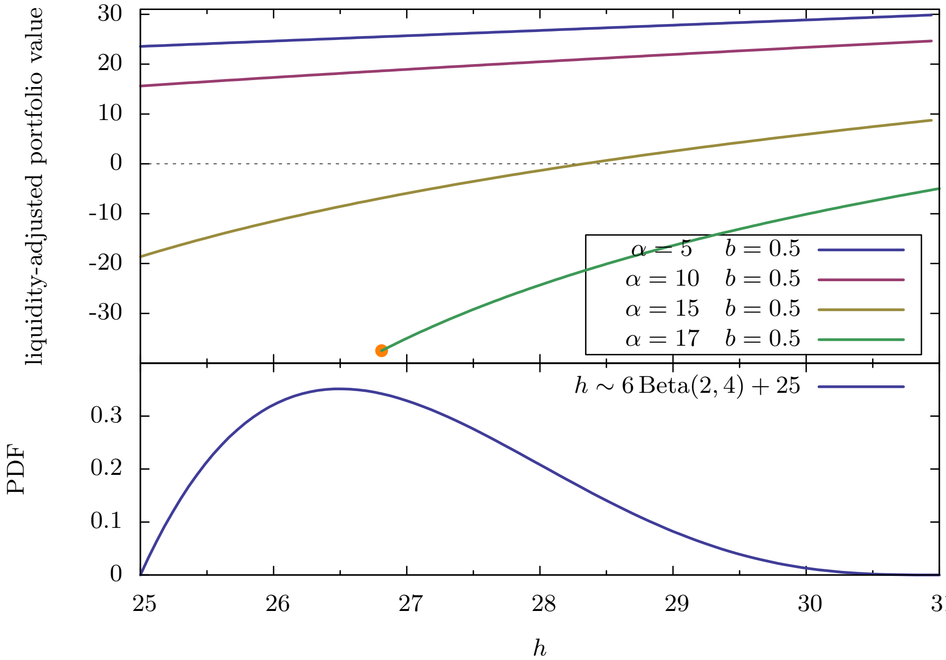

Figure 1: Upper part: Liquidity-adjusted portfolio values as a function of the asset price $h = h_1 = h_2$ for the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ with fixed b and varying α. Lower part: PDF of the asset price h, where $h \sim 6 \cdot \mathrm{Beta}(2,4) + 25$.

UBSR is a distribution-based convex monetary risk measure which is continuous from above and below, see, e.g., Chapter 4.9 in Föllmer & Schied (2011) for a detailed discussion of basic properties, its robust representation, and its relation to expected utility theory. Moreover, it is easy to check that $y = \rho(X)$ is the unique solution of the equation

$$E\left[\ell(-X - y)\right] = z.$$

This implicit characterization reduces the computation of UBSR to a stochastic root finding problem, and it is thus particularly useful for the numerical estimation of the downside risk. Combined with Theorem 3.7 it provides the basis for an efficient algorithm to compute the liquidity-adjusted risk ρ^V.

Corollary 4.2. *Assume that P-almost surely $\bar{\eta} \mapsto V(\bar{\eta})$ is continuous on the interior of its essential domain, and let $\bar{\xi}$ be a portfolio such that P-almost surely $\bar{\xi} + \rho^V(\bar{\xi})$ is in the interior of the essential domain of V. Then $\rho^V(\bar{\xi})$ is equal to the unique root $k^* \in \mathbb{R}$ of the function*

$$g : \mathbb{R} \to \mathbb{R}, \quad k \mapsto E\left[\ell(-V(\bar{\xi} + k)) - z\right]. \tag{6}$$

The complexity of the random supply-demand curves and constraints will typically require a numerical evaluation of the value of the function g at a given argument $k \in \mathbb{R}$. To obtain a solution of the root finding problem described above we use two stochastic approximation algorithms, the Robbins-Monro and the Polyak-Ruppert algorithm. For a detailed analysis of suitable algorithms we refer to Dunkel & Weber (2010). The numerical results are obtained for the following specifications of our market model.

Portfolio construction In order to illustrate the interplay of price effects, limited access to financing, and convex risk measures, we consider a financial market with three assets: cash and two risky assets indexed by $i = 1, 2$. We fix a portfolio $\bar{\xi}$ consisting of zero cash, a short position of three shares of asset $i = 1$, and a long position of four shares of asset $i = 2$, i.e., $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$.

For the purpose of comparative statics, we compare different random supply-demand curves. Specifically, we assume that the financial market of the risky assets ($i = 1, 2$) is characterized by exponential marginal supply-demand curves $m_i(x) = h_i \cdot e^{-bx}$ with $b, h_1, h_2 > 0$. The slope b of the exponent is treated as a model parameter, while h_i, $i = 1, 2$, are modeled as random variables. We compare three values of b: 0.005 (which can essentially be considered as a value of 0), 0.5 and 1. The parameter $b = 0.005$ corresponds to a market with essentially no price impact, $b = 0.5$ corresponds to a medium-size price impact, and $b = 1$ to a large price impact. The stochastic parameters h_i, $i = 1, 2$, have a shifted beta distribution $h_i - s \sim M \cdot \mathrm{Beta}(2, 4)$. For the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ we choose $(s, M) = (25, 6)$. The parameters (s, M) shift and rescale the original beta distribution such that the support of h_i, $i = 1, 2$, equals the interval $[25, 31]$. We compare the results for three different dependence structures of the assets $i = 1$ and $i = 2$: comonotonicity, independence, and countermonotonicity.

Limited access to financing becomes particularly important, if the absolute values of negative short-term cash flows are large. We use proportional margin constraints

$$\phi(\xi) = -\alpha \cdot \xi_1^- - \alpha \cdot \xi_2^-$$

for various values of α. The larger α, the larger is the absolute value of the short-term cash flows, and the more important are the liquidity constraints. The parameter a in Def. 2.9 is set to -0.6.

The last ingredient of our specification are the portfolio constraints. We fix short selling constraints $\mathcal{K} = [-q_i, \infty)^2$ for $q_i \geq 0$, $i = 1, 2$. The values of parameter q_i are set to 4, which prohibits short selling 4 or more assets.

Table 1: Liquidity-adjusted portfolio values $V(\bar{\xi})$ and the corresponding optimal portfolios $(\eta_0^*, \eta_1^*, \eta_2^*)$ as functions of h in the comonotonic case. These results refer to the portfolio $\bar{\xi} = (0, -3, 4)$ and parameters $\alpha \in \{5, 15\}$ and $b = 0.5$.

$\alpha = 5 \quad b = 0.5$				
h	$V(\bar{\xi})$	η_0^*	η_1^*	η_2^*
25	23.55	15.92	-3.30	3.61
26	24.63	15.86	-3.29	3.63
27	25.69	15.80	-3.28	3.64
28	26.76	15.75	-3.27	3.66
29	27.81	15.70	-3.26	3.67
30	28.86	15.66	-3.25	3.69
31	29.91	15.62	-3.24	3.70

$\alpha = 15 \quad b = 0.5$				
h	$V(\bar{\xi})$	η_0^*	η_1^*	η_2^*
25	-18.63	55.95	-3.77	0.78
26	-11.50	55.96	-3.77	1.17
27	-5.92	55.90	-3.76	1.47
28	-1.33	55.78	-3.75	1.71
29	2.54	55.63	-3.74	1.91
30	5.91	55.44	-3.73	2.08
31	8.90	55.24	-3.72	2.22

Liquidity-adjusted portfolio values The liquidity-adjusted portfolio value is a function of the realizations of the random parameters h_i, $i = 1, 2$. In the case of comonotonicity and countermonotonicity, this function effectively depends only on one parameter, since the realization of h_1 is a monotonic function of the realization of h_2.

Figure 1 displays the liquidity-adjusted portfolio value $V(\bar{\xi})$ as a function of $h_1 = h_2 =: h$ in the comonotonic case for the portfolio $(\xi_0, \xi_1, \xi_2) = (0, -3, 4)$. We focus on the parameter values $b = 0.5$ and $\alpha \in \{5, 10, 15, 17\}$. $b = 0.5$ corresponds to medium-size price impact. The size of the short-term cash flows induced by short asset positions increases with α. Increasing

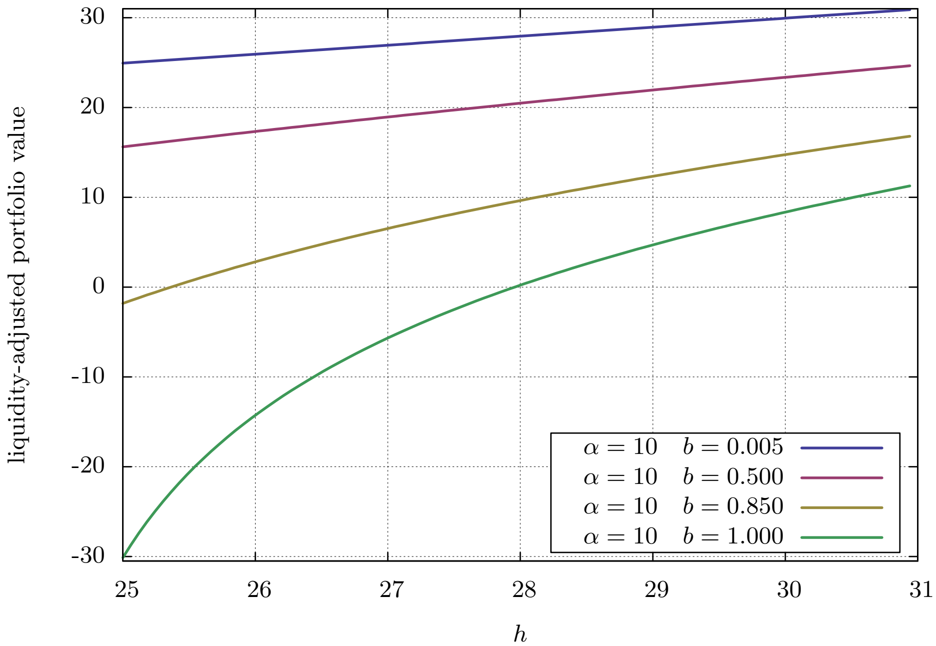

Figure 2: Liquidity-adjusted portfolio value as a function of $h = h_1 = h_2$ for the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ with $\alpha = 10$ and varying b.

α thus leads to lower liquidity-adjusted values of the portfolio. If $\alpha = 17$ – corresponding to particularly high short-term cash flows – the constraints cannot be satisfied anymore for low values of h and a default occurs. In this case, the liquidity-adjusted value equals $-\infty$. In the figure this discontinuity is emphasized by an orange dot.

Comonotonicity implies that increasing h increases both the prices of asset $i = 1$ and asset $i = 2$. For $h \in [25, 31]$ the marginal price for buying or selling the first infinitesimal unit of assets $i = 1, 2$ is at least 25. Short positions are, however, associated with short-term cash flows of α per unit. If $\alpha \in \{5, 10, 15, 17\}$ it turns out to be suboptimal to reduce the size of the short position in asset $i = 1$ by buying shares when computing the optimal portfolio $\bar{\eta}^*$ in equation (1), since $\alpha \leq 25$ in these cases. This is confirmed numerically for portfolios with a finite liquidity-adjusted value showing that $-4 < \eta_1^* < -3$. At the same time, we observe that the investor optimally sells more units of asset $i = 2$ than asset $i = 1$. The rational for this phenomenon is that the short position in asset $i = 1$ incurs an additional temporary cost which is caused by the short-term cash flows.

Both assets are optimally sold, and the liquidity-adjusted value of the portfolio increases with h, the multiplicative factor of the supply-demand curves. For smaller values of h, more assets need to be sold in order to satisfy the liquidity constraint. The average price that can be achieved in this case is smaller, because supply-demand curves are downward sloping for $b = 0.5$. At the same time, the obligation associated with the short position becomes relatively more important, since more assets of the long position are optimally liquidated. This explains why the liquidity-adjusted value of the portfolio is a concave function of h in all cases $\alpha \in \{5, 10, 15, 17\}$. For $\alpha = 5$, i.e. a modest liquidity constraint, the optimal portfolio $\bar{\eta}^*$ in equation (1) is quite insensitive to changes in h. In this case, we have e.g. $\bar{\eta}^* = (-15.9, -3.3, 3.6)$ for $h = 25$ and $\bar{\eta}^* = (-15.6, -3.2, 3.7)$ for $h = 31$. Only a small amount of shares needs to be liquidated. The liquidity-adjusted value is thus an almost linear function of h. The liquidity constraint is stronger

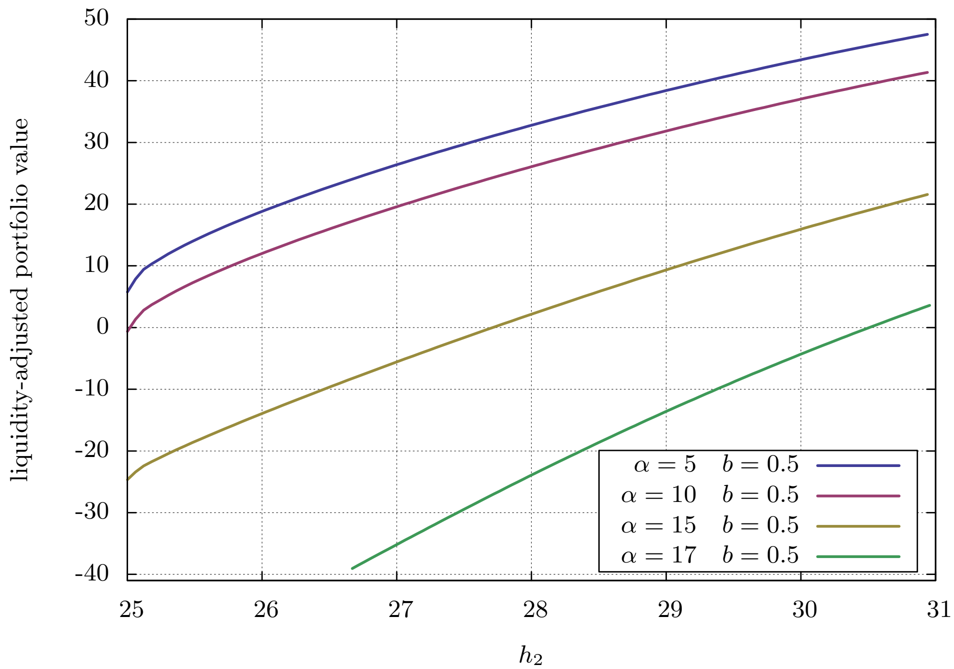

Figure 3: Liquidity-adjusted portfolio value $V(\bar{\xi})$ in the countermonotonic case as a function of h_2 for $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ with fixed b and varying α.

for larger α implying that more shares need to be sold. If h becomes smaller, prices decrease and the amount of shares that is sold needs to be increased. Since average prices decrease with the number of shares sold, even more shares need to be sold in order to fulfill the liquidity constraint. The concavity of the liquidity-adjusted value as a function of h is, hence, more pronounced for larger α. For $\alpha = 15$ we obtain, for example, optimal portfolios $\bar{\eta}^* = (55.9, -3.8, 0.8)$ for $h = 25$ and $\bar{\eta}^* = (55.2, -3.7, 2.2)$ for $h = 31$. Optimal portfolios for further values of α and h are provided in Table 1.

Figure 2 illustrates the liquidity-adjusted portfolio value as functions of the asset price $h = h_1 = h_2$ for fixed $\alpha = 10$ (short-term cash flows) and varying b (price impact). Increasing the price impact b does, of course, decrease the portfolio value. Again, comonotonicity implies that increasing h increases both the prices of asset $i = 1$ and asset $i = 2$. At the same time, it remains suboptimal to reduce the size of the short position in asset $i = 1$ by buying shares, since also in this case short-term cash flows per share $\alpha = 10$ are smaller than the lower bound 25 for the marginal price of buying or selling the first infinitesimal unit of asset $i = 1$. If $b = 0.005$, there is essentially no price impact and the liquidity-adjusted portfolio value is almost linear in h. If b is increased, the price impact becomes larger. Due to a non negligible liquidity constraint for $\alpha = 10$, cash is required and shares of the assets need to be sold. Again, if h is smaller, prices decrease and the amount of shares that needs to be sold is increased. Since average prices decrease with the number of shares sold, even more shares need to be sold in order to fulfill the liquidity constraint. The concavity of the liquidity-adjusted value as a function of h becomes more pronounced for larger price impact b. For $b = 1$ we obtain, for example, optimal portfolios $\bar{\eta}^* = (37.8, -3.8, 1.1)$ for $h = 25$ and $\bar{\eta}^* = (35.6, -3.6, 2.8)$ for $h = 31$.

Table 2 shows the means and variances of the liquidity-adjusted portfolio value for $b \in \{0.005, 0.5, 1\}$ and $\alpha \in \{5, 10, 15, 20\}$. The qualitative behavior of these quantities can already be inferred from Figures 1 and 2, if the distribution of h is given as displayed in the lower part of

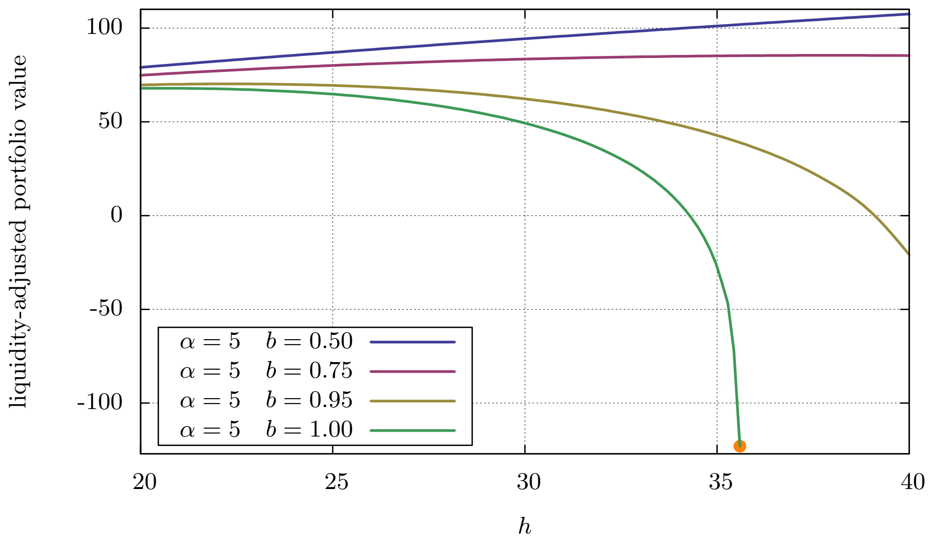

Figure 4: Liquidity-adjusted portfolio value as a function of $h = h_1 = h_2$ for the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (45, -5, 7)$ with fixed α and varying b.

Figure 1: the mean decreases and the variances increase for increasing α (liquidity constraints) and increasing b (price impact).

In contrast to Figure 1 and 2, Figure 3 displays the liquidity-adjusted portfolio value in the countermonotonic case. In this situation, h_1 and h_2 are decreasing functions of each other. We plot the liquidity-adjusted portfolio value as a function of h_2, the multiplicative factor of the supply-demand curve of long asset $i = 2$. If h_2 increases, prices of asset $i = 2$ increase, while prices of the short asset $i = 1$ decrease. Like Figure 1, Figure 3 shows the liquidity-adjusted portfolio value for $b = 0.5$ and $\alpha \in \{5, 10, 15, 17\}$. Qualitatively, the findings are very similar in both cases. Again, it is optimal to liquidate shares of both assets. The main difference is that the range of liquidity-adjusted value of the portfolio changes for varying h_2 is significantly larger in the countermonotonic case. For large h_2 (associated with a small h_1) the long position in η_2^* is valuable, but the absolute value of the negative mark-to-market η_1^* is smaller than in comonotonic case. An analogous argument applies, if h_2 is small. In this case the long position has a smaller value than for large h_2, while the short position constitutes a larger obligation.

For further comparison, we consider also a different portfolio which contains 45 units of cash, a short position of five shares of asset $i = 1$ and a long position of seven shares of asset $i = 2$, i.e. $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (45, -5, 7)$. Corresponding to comonotonic dependence, we assume that $h = h_1 = h_2$. In contrast to the other example, we suppose that the range of h is given by $[20, 40]$. In this case, the short selling constraint requires the investor to buy at least one share of asset $i = 1$. A similar effect would also occur in other examples, if α was large compared to h_1.

Figure 4 displays the liquidity-adjusted portfolio value of the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (45, -5, 7)$ as a function of $h = h_1 = h_2$ for fixed α (short-term cash flows) and varying b (price impact). The liquidity-adjusted value does, of course, decrease with increasing price effect b. We observe that the liquidity-adjusted value is not necessarily a monotonically increasing function. For a market with a medium size price effect ($b = 0.5$), the liquidity-adjusted portfolio value still increases with increasing h, while it decreases for large h in the case of large price impacts ($b \in \{0.95, 1\}$) . This can be understood by inspecting the optimal portfolios $\bar{\eta}^*$. In all cases, both the liquidity constraint and the short selling constraint are binding. In order to satisfy the short selling constraint the investor buys exactly one share of asset $i = 1$. The price of this share increases with h and needs to be financed by selling asset $i = 2$. For $b = 0.95$ the position in asset $i = 2$ needs to be reduced from 7 shares to 6.5 shares if $h = 20$, but to 3 shares if $h = 40$. This diminishes the liquidity-adjusted portfolio value.

Liquidity-adjusted risk measures In this section we focus again on the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ and assume that the stochastic parameters h_i, $i = 1, 2$, have a shifted beta distribution $h_i - 25 \sim 6 \cdot \mathrm{Beta}(2, 4)$. In all cases we ran $n = 5000$ independent simulations yielding an empirical distribution of liquidity-adjusted portfolio values or liquidity-adjusted risk measures. The individual samples of the portfolio values are solutions to the optimization problem described in Section 2.3. We computed moments of the empirical distributions. We estimated liquidity-adjusted VaR at level 5%, AVaR at level 5% as well as utility-based shortfall risk (UBSR) with an *exponential loss function* $\ell_{\exp}(x) = \exp(0.5x)$ and with threshold level $z = 0.05$. All risk measures were computed both according to the approach of Acerbi & Scandolo (2008) and according to our approach, see Definition 3.4 and equation (6); the estimates were labeled by (AS) and (V), respectively. The results are documented in Tables 2 (h_1 and h_2 comonotonic), 3 (h_1 and h_2 independent) and 4 (h_1 and h_2 countermonotonic).

The liquidity-adjusted portfolio values have generally lower variance in the comonotonic case than in the countermonotonic case. The independent case exhibits intermediate values. The mean of the liquidity-adjusted value always decreases with larger short-term cash flows and larger price impact.

Tables 2, 3 and 4 demonstrate that all risk measures detect the increase of liquidity risk as b (price impact) and α (short term cash flows/liquidity constraints) increase. As proven in Proposition 3.5, the absolute value of our liquidity-adjusted risk measures is indeed always smaller than the one suggested by Acerbi & Scandolo (2008). Furthermore, in some cases the liquidity-adjusted risk measure according to Acerbi & Scandolo (2008) becomes infinite, while our liquidity-adjusted risk measure is still finite. The reason is that our risk-measure computes the cash amount that needs to be added to the position at time 0 in order to make it acceptable. The position can thus still be modified such that default is prevented. As explained in Section 3.2, the approach of Acerbi & Scandolo (2008) computes the amount of cash that needs to be added to the liquidity-adjusted value at time 1 to make it acceptable. In the event of default, this amount will be infinitely large. The *ex post* inflow of cash cannot prevent a default once it has occurred.

When conducting our numerical experiments we also noticed that variance reduction techniques become important when computing liquidity-adjusted risk measures if price effects and short-term cash flows are large. Suitable techniques are described in Dunkel & Weber (2007) and Dunkel & Weber (2010). The effective implementation in the context of measuring risk in highly illiquid markets constitutes an interesting direction for future research.

5 Conclusion

In the current paper we propose liquidity-adjusted risk measures in the context of a static one-period model. Main drivers are two dimensions of liquidity risk, namely price impact of trades and limited access to financing. The suggested cash-invariant risk measures are based on the notion of capital requirements and provide a simple method to properly managing portfolio risk by injecting an appropriate amount of capital upfront. Our analysis is based on the notion of liquidity-adjusted portfolio valuation that was originally developed by Acerbi & Scandolo (2008).

Our approach is quite stylized, and it remains an important topic for future research to investigate how liquidity-adjusted valuation and risk measurement can successfully be implemented in practice. In particular, the random supply-demand curves and the liquidity constraints of the model would have to constitute appropriate proxies of reality. This requires the design and detailed analysis of suitable estimation procedures.

Two further topics are important and might be promising for future research. First, a dynamic extension of the current framework could provide a more realistic approach to measuring liquidity-adjusted risk. Second, liquidity-adjusted risk measures might contribute to the theory of portfolio choice. Modified objective functions or constraints that integrate the results of this paper will lead to different optimal investments which do not ignore the important dimension of liquidity risk anymore.

Table 2: Liquidity-adjusted risk measures for the portfolio $\bar{\xi} = (0, -3, 4)$ in the comonotonic case $h = h_1 = h_2$.

		b=0.005							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	27.0	1.1	-25.4	-25.5	-25.2	-25.3	-20.7	-20.8
	10	27.0	1.1	-25.2	-25.4	-25.1	-25.3	-20.6	-20.7
	15	26.9	1.1	-25.1	-25.3	-24.9	-25.2	-20.4	-20.6
	20	26.7	1.1	-24.8	-25.2	-24.6	-25.0	-20.2	-20.4
		b=0.5							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	25.7	1.2	-17.1	-24.1	-17.0	-23.9	-14.5	-19.4
	10	18.9	2.7	-8.3	-16.5	-7.4	-16.2	-6.6	-12.3
	15	-6.4	27.7	3.9	14.9	4.4	16.2	4.7	17.7
	20	$-\infty$	∞	17.9	∞	18.0	∞	54.2	∞
		b=1							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	24.0	1.5	-11.1	-22.2	-10.2	-22.0	-9.6	-17.7
	10	-7.0	59.5	2.7	20.8	3.1	23.7	3.1	25.7
	15	$-\infty$	∞	18.5	∞	18.5	∞	18.7	∞
	20	$-\infty$	∞	34.3	∞	34.4	∞	41.7	∞

Table 3: Liquidity-adjusted risk measures for the portfolio $\bar{\xi} = (0, -3, 4)$ in the independent case.

		b=0.005							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	27.1	28.9	-17.7	-18.4	-16.1	-16.5	-14.5	-14.6
	10	27.0	28.9	-17.3	-18.4	-15.9	-16.5	-14.4	-14.5
	15	26.9	28.9	-17.2	-18.3	-15.6	-16.4	-14.3	-14.4
	20	26.8	28.9	-17.1	-18.1	-15.6	-16.2	-14.1	-14.3
		b=0.5							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	25.8	28.9	-12.3	-17.1	-11.0	-15.2	-10.3	-13.3
	10	19.0	30.1	-5.1	-10.4	-4.0	-8.5	-3.7	-6.6
	15	-6.0	51.1	5.2	16.6	6.5	18.1	6.3	19.6
	20	$-\infty$	∞	18.3	∞	30.2	∞	21.1	∞
		b=1							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	24.1	29.1	-8.1	-15.4	-7.0	-13.6	-6.8	-11.6
	10	-6.2	64.3	3.2	18.6	4.6	20.6	4.0	23.0
	15	$-\infty$	∞	18.3	∞	20.0	∞	18.7	∞
	20	$-\infty$	∞	34.1	∞	36.7	∞	34.4	∞

Table 4: Liquidity-adjusted risk measures for the portfolio $\bar{\xi} = (0, -3, 4)$ in the countermonotonic case.

		b=0.005							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	26.8	54.2	-14.9	-14.9	-13.2	-13.1	-12.3	-12.3
	10	26.8	54.2	-14.8	-14.9	-13.1	-13.0	-12.2	-12.2
	15	26.7	54.2	-14.7	-14.8	-13.0	-13.0	-12.0	-12.1
	20	26.5	54.3	-14.5	-14.6	-12.8	-12.8	-11.8	-12.0
		b=0.5							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	25.5	54.2	-11.2	-13.6	-8.7	-11.8	-12.3	-12.3
	10	18.8	55.0	-4.4	-6.9	-1.8	-5.1	-12.2	-12.2
	15	-6.0	71.2	6.0	18.8	7.6	20.4	7.0	21.2
	20	$-\infty$	∞	18.8	∞	28.0	∞	21.1	∞
		b=1							
		mean	variance	VaR(V)	VaR(AS)	AVaR(V)	AVaR(AS)	UBSR(V)	UBSR(AS)
α	5	23.8	54.3	-7.4	-12.0	-5.1	-10.2	-5.7	-9.3
	10	-5.8	67.8	3.8	17.8	5.2	19.1	4.5	20.1
	15	$-\infty$	∞	18.2	∞	19.5	∞	18.6	∞
	20	$-\infty$	∞	34.0	∞	34.6	∞	34.2	∞

A Proofs

Proof of Prop. 2.18. The proofs of (1), (3) and (4) are similar to Proposition 3 and Theorem 1 in Acerbi & Scandolo (2008).

(2) can be shown as follows: If $L(\bar{\xi}) \geq a$, then $(L(\bar{\xi}), 0_N) \in \mathcal{L}^{\bar{\xi}}$, since $0_N \in \mathcal{K}$. Thus, $V(\bar{\xi}) \geq L(\bar{\xi}) \geq a > -\infty$.

In order to verify (5), suppose that $\bar{\xi} \leq \bar{\eta}$ and recall that attainable portfolios $\bar{\mu} \in \mathcal{A}(\bar{\xi}, \bar{m})$, $\bar{\nu} \in \mathcal{A}(\bar{\eta}, \bar{m})$ take the form

$$\bar{\mu} = \left(\xi_0 + \sum_{i=1}^{N} \int_0^{\alpha_i} m_i(x)dx, \xi - \alpha\right), \quad \bar{\nu} = \left(\eta_0 + \sum_{i=1}^{N} \int_0^{\beta_i} m_i(x)dx, \eta - \beta\right) \quad (\alpha, \beta \in \mathbb{R}^N).$$

We associate to any $\bar{\mu} \in \mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$ the vector $\bar{\nu}$ corresponding to $\beta = \eta - \xi + \alpha \geq \alpha$, i.e.

$$\bar{\nu} = \left(\eta_0 + \sum_{i=1}^{N} \int_0^{\eta_i - \xi_i + \alpha_i} m_i(x)dx, \xi - \alpha\right).$$

Then $\bar{\nu}$ belongs to $\mathcal{L}(\bar{\eta}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$, and we have the inequality $U(\bar{\mu}, \bar{m}) \leq U(\bar{\nu}, \bar{m})$. This implies $V(\bar{\xi}, \bar{m}) \leq V(\bar{\eta}, \bar{m})$. □

Proof of Prop. 3.5. Since $L(\bar{\xi}, \bar{m}) \in L^\infty$, there exists $k \in \mathbb{R}$ such that both $L(k + \bar{\xi}, \bar{m}) = k + L(\bar{\xi}, \bar{m}) \geq a$ and $L(k + \bar{\xi}, \bar{m}) = k + L(\bar{\xi}, \bar{m}) \in \mathcal{A}$. Thus, by Prop. 2.18, $L(k + \bar{\xi}, \bar{m}) \leq V(k + \bar{\xi}) \leq U(k + \bar{\xi}, \bar{m}) \in L^\infty$. Hence, $V(k + \bar{\xi}) \in \mathcal{A}$. This implies that $\rho^V(\bar{\xi}) < \infty$.

Moreover, observe that $V(k + \bar{\xi}) \leq U(k + \bar{\xi}, \bar{m}) = k + U(\bar{\xi}, \bar{m})$. Thus,

$$\rho^V(\bar{\xi}) = \inf\{k : V(k + \bar{\xi}) \in \mathcal{A}\} \geq \inf\{k : U(k + \bar{\xi}, \bar{m}) \in \mathcal{A}\} = \rho(U(\bar{\xi}, \bar{m})) > -\infty,$$

since $U(\bar{\xi}, \bar{m}) \in L^\infty$.

Estimate (3) is a consequence of the translation-supervariance (2) of $V(\bar{\xi})$. Indeed, if $\rho^V(\bar{\xi}) > 0$, then we have $V(k + \bar{\xi}) \notin \mathcal{A}$ for any fixed $k \in (0, \rho^V(\bar{\xi}))$. Since $V(k + \bar{\xi}) \geq k + V(\bar{\xi})$, this yields $k + V(\bar{\xi}) \notin \mathcal{A}$, hence $\rho^{\mathrm{AS}}(\bar{\xi}) \geq k > 0$. Letting k increase to $\rho^V(\bar{\xi})$, we obtain $\rho^V(\bar{\xi}) \leq \rho^{\mathrm{AS}}(\bar{\xi})$ for $\rho^V(\bar{\xi}) > 0$. Conversely, $\rho^V(\bar{\xi}) < 0$ implies that $V(k + \bar{\xi}) \in \mathcal{A}$ for any fixed $k \in (\rho^V(\bar{\xi}), 0)$. Here translation-supervariance yields the estimate $V(k+\bar{\xi}) \le k + V(\bar{\xi})$. Thus, $k + V(\bar{\xi}) \in \mathcal{A}$, and so we have $\rho^{\mathrm{AS}}(\bar{\xi}) \le k < 0$. Taking the limit $k \downarrow \rho^V(\bar{\xi})$, this translates into $\rho^{\mathrm{AS}}(\bar{\xi}) \le \rho^V(\bar{\xi})$ for $\rho^V(\bar{\xi}) < 0$.

$\square$

Proof of Theorem 3.6. Letting $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{N+1}$ and $m \in \mathbb{R}$, we obtain that

$$\rho^V(\bar{\xi} + m) = \inf\{k : V(k + \bar{\xi} + m) \in \mathcal{A}\} = \rho^V(\bar{\xi}) - m,$$

which proves the cash-invariance of ρ^V. Suppose now that $\bar{\xi} \le \bar{\eta}$. Then $V(k+\bar{\xi}) \le V(k+\bar{\eta})$ for any $k \in \mathbb{R}$. Thus,

$$V(k+\bar{\xi}) \in \mathcal{A} \quad \Rightarrow \quad V(k+\bar{\eta}) \in \mathcal{A},$$

since $\mathcal{A}$ is the acceptance set of the risk measure ρ. Hence, $\rho^V(\bar{\eta}) \le \rho^V(\bar{\xi})$.

In order to prove convexity, we fix $\alpha \in [0,1]$ and $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{N+1}$. For all $k_1, k_2 \in \mathbb{R}$ such that $V(k_1 + \bar{\xi}), V(k_2 + \bar{\eta}) \in \mathcal{A}$, convexity of the acceptance set $\mathcal{A}$ yields that $\alpha V(k_1 + \bar{\xi}) + (1-\alpha)V(k_2 + \bar{\eta}) \in \mathcal{A}$. Since V is concave by Prop. 2.18, we have $\alpha V(k_1 + \bar{\xi}) + (1-\alpha)V(k_2 + \bar{\eta}) \le V(\alpha(k_1 + \bar{\xi}) + (1-\alpha)(k_2 + \bar{\eta}))$, hence

$$V(\alpha k_1 + (1-\alpha)k_2 + \alpha\bar{\xi} + (1-\alpha)\bar{\eta}) \in \mathcal{A}.$$

This implies $\alpha k_1 + (1-\alpha)k_2 \ge \rho^V(\alpha\bar{\xi} + (1-\alpha)\bar{\eta})$. Taking the limits $k_1 \downarrow \rho^V(\bar{\xi})$ and $k_2 \downarrow \rho^V(\bar{\eta})$, we obtain convexity of ρ^V. $\square$

Proof of Theorem 3.7. Let $\rho^V(\bar{\xi}) = k$. The cash invariance of ρ^V implies that

$$0 = \rho^V(\bar{\xi} + k) = \inf\{m : V(\bar{\xi} + k + m) \in \mathcal{A}\}. \tag{7}$$

Since V is increasing in the portfolio and $\mathcal{A}$ is an acceptance set, we have $V(\bar{\xi} + k + m) \in \mathcal{A}$ for all $m > 0$. Thus, $\rho(V(\bar{\xi} + k)) = \lim_{m \searrow 0} \rho(V(\bar{\xi} + k + m)) \le 0$.

Suppose that $\rho(V(\bar{\xi} + k)) < -\epsilon < 0$ for $\epsilon > 0$, i.e. $\rho(V(\bar{\xi} + k) - \epsilon) < 0$. Since V is P-almost surely continuous on the interior of its essential domain and increasing in the portfolio vector, there exists $k' < k$ such that $V(\bar{\xi} + k') \ge V(\bar{\xi} + k) - \epsilon$. By inverse monotonicity of the risk measure ρ, we get that $\rho(V(\bar{\xi} + k')) \le \rho(V(\bar{\xi} + k) - \epsilon) < 0$, i.e. $V(\bar{\xi} + k') \in \mathcal{A}$ – contradicting (7). Thus, $\rho(V(\bar{\xi} + k)) = 0$.

Uniqueness of the solution of (4) can be shown as follows: suppose there exists two solutions $k' > k$ to the equation. Letting $k' - k =: c > 0$, we get

$$\rho(V(\bar{\xi} + k')) = \rho(V(\bar{\xi} + k + c)) \le \rho(V(\bar{\xi} + k) + c) = \rho(V(\bar{\xi} + k)) - c = -c < 0,$$

a contradiction. Here, the inequality follows from the translation-supervariance of V, see Theorem 2.18. $\square$

Proof of Prop. 3.10. Let $\bar{\xi} \in \mathbb{R}^{N+1}$ be a given portfolio and $k \in \mathbb{R}$ the units of the eligible asset e_ϵ with $\epsilon > 0$. We are going to show that

$$V(\bar{\xi}) + k \le \tilde{V}((\bar{\xi}, k)) \le V(\bar{\xi} + \epsilon) + k + \epsilon \quad \text{for all } k \in \mathbb{R}, \epsilon > 0, \tag{8}$$

where $\tilde{V}$ depends on ϵ implicitly. Indeed, for any $\bar{\eta} = (\eta_0, \eta) \in \mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$ the portfolio

$$\tilde{\mu} := \left(\eta_0 + \int_0^\epsilon e_\epsilon(x)\, dx, \eta, k - \epsilon\right) \in \mathbb{R}^{N+2}.$$

is attainable from $(\bar{\xi}, k)$, belongs to $\tilde{\mathcal{L}}((\bar{\xi}, k), (\bar{m}, e_\epsilon), \tilde{\phi}, a) \cap (\mathbb{R} \times \tilde{\mathcal{K}})$ and satisfies $\tilde{U}(\tilde{\mu}, (\bar{m}, e_\epsilon)) = U(\bar{\eta}, \bar{m}) + k$. This yields the first inequality $V(\bar{\xi}) + k \leq \tilde{V}((\bar{\xi}, k))$.

In order to verify the second inequality in (8), note first that buying more as well as selling more than ϵ units of the eligible asset decreases $\tilde{U}(\cdot, (\bar{m}, e_\epsilon))$. For the relevant portfolios $\tilde{\eta} = (\bar{\eta}, \eta_{N+1}) \in \mathcal{L}((\bar{\xi}, k), (\bar{m}, e_\epsilon), \phi, a) \cap (\mathbb{R} \times \mathcal{K} \times [k - \epsilon, k + \epsilon])$, we have $\tilde{U}(\tilde{\eta}, (\bar{m}, e_\epsilon)) = U(\bar{\eta}, \bar{m}) + \eta_{N+1} \leq U(\bar{\eta}, \bar{m}) + k + \epsilon$. Note that $\bar{\eta}$ is an element of $\mathcal{L}(\bar{\xi} + \delta, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$ for some $\delta \in [-\epsilon, \epsilon]$. This implies

$$\tilde{V}((\bar{\xi}, k)) \leq V(\bar{\xi} + \delta) + k + \epsilon \leq V(\bar{\xi} + \epsilon) + k + \epsilon,$$

and so we have shown (8).

The inequality (8) translates into

$$\rho(V(\bar{\xi})) \geq \rho^{e_\epsilon}(\bar{\xi}) \geq \rho(V(\bar{\xi} + \epsilon)) - \epsilon.$$

Since ρ is continuous and V is continuous at $\bar{\xi}$, letting ϵ tend to 0 yields

$$\lim_{\epsilon \downarrow 0} \rho^{e_\epsilon}(\bar{\xi}) = \rho(V(\bar{\xi})) = \rho^{\mathrm{AS}}(\bar{\xi}).$$

□

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